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SUPERMODULAR LATTICES

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CONTENTS

DEDICATION

Mr. Mohammed Jalauddin Kokan
and
Mrs. Aaisha Begum

We dedicate this book to the memory of Dr. Iqbal Unnisa's parents. Her father Mr. Mohammed Jalauddin Kokan was the Post Master in Meenambur and mother Mrs. Aaisha Begum was a housewife. From this remote village of Meenambur in Tamilnadu, her parents shifted to Chennai to educate their children.

In those days, girl children being enrolled in primary school was rare. Yet, the Kokans succeeded in educating their daughters into doctors and doctorates. In fact, Dr. Iqbal Unnisa has the distinction of being the first Muslim women in India to receive her doctoral degree in Mathematics.

This dedication is a gesture of honouring Mr. Mohammed Jalauddin Kokan and Mrs. Aisha Begum, and at the same time, acknowledging the great role that education plays in the empowerment of women.

PREFACE

In lattice theory the two well known equational class of lattices are the distributive lattices and the modular lattices. All distributive lattices are modular however a modular lattice in general is not distributive.

In this book, new classes of lattices called supermodular lattices and semi-supermodular lattices are introduced and characterized as follows:

A subdirectly irreducible supermodular lattice is isomorphic to the two element chain lattice C_2 or the five element modular lattice M_3.

A lattice L is supermodular if and only if L is a subdirect union of a two element chain C_2 and the five element modular lattice M_3.

A modular lattice L is n-semi-supermodular if and only if there does not exist a set of (n + 1) elements $a_1, a_2, \ldots, a_n$ in L such that $a + a_1 = a + a_2 = \ldots = a + a_n > a$ with $a > a_ia_j$ (i not equal j); i, j = 1, 2, …, n.

A modular lattice L is n-semi-supermodular if and only if it does not contain a sublattice whose homomorphic images is isomorphic to $M_{i_1+2,i_2+2,\ldots,i_r+2}$ or $\widehat{M}_{i_1+2,i_2+2,\ldots,i_r+2}$ with $i_1 + i_2 + \ldots + i_r = n - 1$; i, j ≥ 1.

We define the concept of Smarandache lattices and GB-algebraic structures are characterized in chapters six and seven respectively.

This book has seven chapters. Chapter one is introductory in nature. A simple modular lattice of finite length is introduced and characterized in chapter three. In chapter four the notion of supermodular lattices is introduced and characterized and chapter five introduces the notion of n-semi-supermodular lattices and characterizes them.

It is pertinent to keep on record part of this book is the second authors Ph.D thesis done under the able guidance of the first author Late Professor Iqbalunnisa. Infact the last two authors where planning for a lattice theory book a year back but due to other constraints we could not achieve it. Now we with a heavy heart have made this possible.

We thank Dr. K.Kandasamy for proof reading and being extremely supportive.

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Chapter One

PRELIMINARIES

In this chapter we recall some definitions and results which are made use of throughout this book. The symbols $\leq$, $\geq$, $+$, . will denote inclusion, noninclusion, sum (least upper bound) and product (greater lower bound) in any lattice L; while the symbols, $\subseteq$, $\cup$, $\cap$, $\in$, $\notin$ will refer to set inclusion, union (set sum), intersection (set product), membership, and non-membership respectively. Small letters a, b, … will denote elements of the lattice and greek letters, θ, ϕ, … will stand for congruences on the lattice.

A binary relation θ on L is said to be an equivalence relation if it satisfies.

(i) $x \equiv x\ (\theta)$ (reflexive)

(ii) $x \equiv y\ (\theta) \Rightarrow y \equiv x\ (\theta)$ symmetric

(iii) $x \equiv y\ (\theta)$; $y \equiv x\ (\theta) \Rightarrow x \equiv z\ (\theta)$ (transitive)
If it further satisfies the substitution property.

(iv) $x \equiv x'\ (\theta)$; $y \equiv y'\ (\theta) \Rightarrow x + y \equiv x' + y'\ (\theta)$ then it is called an additive congruence. An equivalence relation which has the substitution property.

(v) $x \equiv x'\ (\theta)$; $y \equiv y'\ (\theta) \Rightarrow xy \equiv x'y'\ (\theta)$ is called a multiplicative congruence. If a binary relation satisfies the conditions (1) to (v) then it is said to be a lattice congruence or merely a congruence on L.

Result 1.1: If a = b (θ) in a lattice L then x = y (θ) for all x, y in L such that ab ≤ x, y ≤ a + b [2].

Let a, b in L such that a ≥ b. Then the set of all elements x in L such that a ≥ x ≥ b is called the interval (a, b). If a = b(a, b) is called a prime interval. If a = b (a, b) is called a trivial interval.

If a ≡ b (θ) (a, b in L; a ≥ b) then x ≡ y (θ) for all x, y in L such that a ≥ x, y ≥ b (by result 1.1) and θ is said to annul the interval (a, b).

Intervals of the form (x, x+y) and (xy, y) are said to be perspective intervals. We say

(xy, y) $\nearrow$ (x, x+y) or equivalently (x, x+y) $\searrow$ (xy, y).

If I = (a, b) is an interval of L then the interval (a+x, b+x) for any x in L is called an additive translate of the interval I and is written as I+x; and the interval (ax, bx) for any x in L is called a multiplicative translate of the interval I and is written as Ix.

An interval J is the lattice translate of an interval I of L if elements $x_1, x_2, \ldots, x_n$ can be found such that

$J = ((((I + x_1)x_2) + x_3)\ldots)\, x_n$ or

$J = ((((I.x_1) + x_2)\, x_3)\ldots)\, x_n$,

where n in finite and +, . occur alternatively.

Result 1.2: Lattice translation is a transitive relation.

Result 1.3: Any lattice translate of an interval I of a distributive lattice L can be written as (I + y)x or (I.x) + y for some x, y in L [10].

Result 1.4: Any subinterval J of an interval I is a lattice translate of I [10].

Result 1.5: Any conqruence θ on L which annuls I annuls all lattice translates of I [10].

Result 1.6: Any non-null set S in L is a congruence class under some congruence relation on L if any only if

(i) S is a convex sublattice and
(ii) Lattice translate of intervals in S lie wholly within S or outside S [10].

DEFINITION 1.1: *The smallest congruence which annuls a set S of a lattice L is called the congruence generated by the set S.*

Result 1.7: Let I be an interval of a lattice L. θ_I the congruence generated by I in L. $x \equiv y\ (\theta_I)$ if and only if there exists a finite set of elements $x+y = x_0 > x_1 > x_2 \ldots .\ x_n = xy$ such that (x_i, x_{i+1}) is a lattice translate of the interval I [10].

Result 1.8: The lattice translate of a prime interval in a modular lattice can only be a prime interval [10].

Result 1.9: Let L be a modular lattice. I and J intervals of L such that I is a lattice translate of J then I is projective with a subinterval of J [10].

DEFINITION 1.2: *The modular lattice consisting of n+2 elements ($n \geq 3$), a, $x_1, \ldots, x_n$, b satisfying $x_i + x_j = a$ ($i \neq j$, $i, j = 1, 2, \ldots, n$), $x_i x_j = b$ for all $i, j = 1, 2, \ldots, n$; $i \neq n$ is denoted by M_n. The modular lattice consisting of the elements a, x_i, $x_2, \ldots, x_n$, b, $y_1, \ldots, y_{m-1}$, c satisfying*

$$x_i + x_j = a \text{ for all } i \neq j,\ i, j = 1, 2, \ldots, n$$

$x_ix_j = b$ for all $i, n \neq j$, $i, j = 1, 2, \ldots, n$
$b+y_i = x_n$ for all $i = 1, 2, \ldots, m-1$
$y_i + y_j = x_n$ for all $i \neq j$, $i, j = 1, 2, \ldots, m-1$
$by_i = c$ for all $i=1, 2, \ldots, m-1$
$y_iy_j = c$ for all $i, j = 1, 2, \ldots, m-1$ is denoted by $M_{n,m}$.

We denote $M_{n_1, n_2, \ldots, n_k}$ $(n_k \geq 3)$ *in a similar fashion.*

We define $\hat{M}_{m,n,r}$ *to be the modular lattice consisting of the elements* $a, x_1, \ldots, x_n, b, y_1, \ldots, y_{m-1}, c, z_1, z_2, \ldots, z_{r-1}, d, cd$ *such that* $a, x_1, \ldots, x_n, b$ *form a lattice isomorphic to* M_n. $x_1, b, y_1, \ldots, y_{m-1}, c$ *form a lattice isomorphic to* M_m; $x_2, b, z_1, \ldots, z_{r-1}, d$ *form a lattice isomorphic to* M_r. *We extend definition to* $\hat{M}_{n,m,r,s}$ *in two ways.*

First $\hat{M}_{n,m,r,s}$ *is got by taking elements* $a, x_1, x_2, \ldots, b, y_1, \ldots, y_{m-1}, b_1, z_1, \ldots, z_{r-1}, b_2, u_1, \ldots, u_{s-1}, b_3, b_1 b_2, b_1b_3, b_3b_1, b_1b_2b_3$ *such that* $a, x_1, x_2, \ldots, x_n, b$ *form a lattice isomorphic to* M_n; $x_1, b, y_1, \ldots, y_{m-1}, b_1$ *form a lattice isomorphic to* M_m, $x_2, b, z_1, \ldots, z_{r-1}, b_2$ *form a lattice isomorphic to* M_r; $x_3, b, u_1, \ldots, u_{s-1}, b_3$ *form a lattice isomorphic to* M_s.

In the second; we define $\hat{M}_{m,n,r,s}$ *by taking the elements*

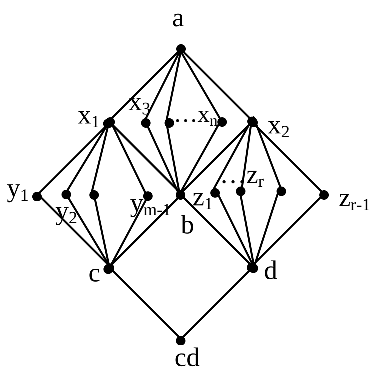
$a, x_1, \ldots, x_n, b, y_1, \ldots, y_{m-1}, b_1, z_1, z_2, \ldots, z_{r-1}, b_2, u_1, \ldots, u_{s-1}, b_3\ b_1b_2, b_2b_3$ *such that* $a, x_1, \ldots, x_n, b$ *form a sublattice isomorphic to* M_n,

$x_1, b, y_1, \ldots, y_{m-1}, b_1$, *form a sublattice isomorphic to* M_m,

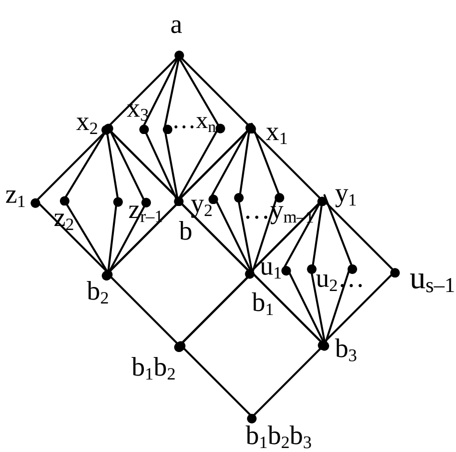
$x_2, b, z_1, z_2, \ldots, z_{r-1}, b_2$, *form a sublattice isomorphic to* M_r,

$y_1, b_1, u_1, \ldots, u_{s-1}, b_3$ *form a sublattice isomorphic to* M_s.

$\breve{M}_{m,n,r,s}$ … *is defined dually.*



$\hat{M}_{m,n,r}$ **Figure 1.1**



$\hat{M}_{n,m,r,s}$ **Figure 1.2**

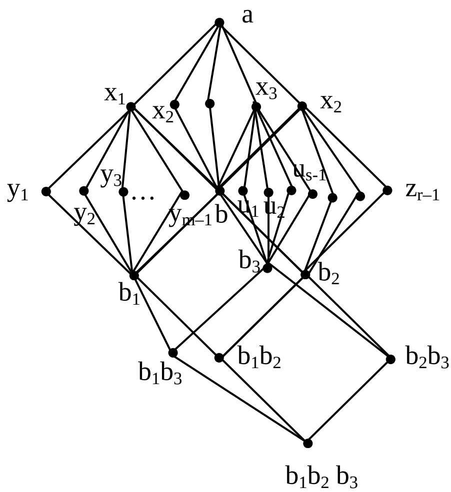


Figure 1.3

These are illustrated in figures 1.1, 1.2 and 1.3.

DEFINITION 1.3: *Let θ be a congruence on a lattice L. θ is called separable if and only if for every pair of comparable elements a < b there exist a finite sequence of elements.*

$$a = x_0 < x_1 < \ldots < x_n = b$$

such that either (x_{i-1}, x_i) *is annulled by θ or* (x_{i-1}, x_i) *consist of single point congruence classes under θ.*

THEOREM [4]: *Let L be a modular lattice and let* C_0 *and* C_I *be chains in L. The sublattice of L generated by* C_0 *and* C_I *is distributive.*

THEOREM [6, 7, 13]: *Any subdirectly irreducible modular lattice of length n ≥ 3 has a sublattice whose homomorphic image is isomorphic to* $M_{3.3}$.

Chapter Two

SIMPLE MODULAR LATTICES OF FINITE LENGTH

In the first chapter we defined the notions of additive translate, multiplicative translate and lattice translate of an interval. We introduce in this chapter, the notion of a distributive translate of an interval and study the properties of distributive translates of prime intervals in a modular lattice. This study leads us to a characterization of simple modular lattices of finite length.

Throughout this chapter L will denote a modular lattice, unless otherwise stated.

DEFINITION 2.1: *A prime interval I of L is said to be distributive if for all nontrivial intervals J = (I+x)y there exist p, q* $\in$ *L with J = Ip + q for all the nontrivial intervals* $J_1 = Ix_1 + y_1$ *there exist* p_1, q_1 *in L with* $J_1 = (1 + p_1)q_1$.

Lemma 2.1: Any prime interval of a distributive lattice is distributive.

***Proof*:** Follows as $J = (I+x)\, y$ implies $J = Iy + xy$ and $J_1 = Ix_1 + y_1$ implies $J_1 = (I+y_1)\,(x_1 + y_1)$.

Lemma 2.2: In a distributive lattice if a nontrivial interval $J = (I+x)y$, then Iy is nontrivial.

***Proof*:** Now $J = Iy + xy$ and if Iy were trivial, so would be J.

DEFINITION 2.2: *If $I = (a, b)$ and $J = (c, d)$ then the intervals $(a + c, b + d)$ and (ac, bd) are denoted by $I + J$ and IJ respectively*.

Lemma 2.3: If a nontrivial interval $J = (I + x)y$ then $I + J$ cannot be trivial.

Proof: Let $I = (a, b)$ and $J = (c, d)$ then
$c = (a + x)y$; $d = (b + x)y$.

Also $a+c = a + (a+x)y = (a+x)\,(a+y)$ and
$b+d = b+(b+x)y = (b+x)\,(b+y)$.

Now if $I + J$ is trivial then $a + c = b + d$ that is
$(a+x)\,(a+y) = (b+x)\,(b+y)$ which means

$$\begin{aligned} a + x &= (a + x) + (a + x)\,(a + y) \\ &= (a + x) + (b + x)\,(b + y) \\ &= (b + x)\,(a + x + b + y) \\ &= b + x. \end{aligned}$$

So $(a + x)y = (b+x)y$ that is J is trivial; a contradiction.
Dually we have

Lemma 2.4: If any nontrivial $J = Ix + y$ then IJ cannot be trivial.

Lemma 2.5: If $I = (a, b)$ is a prime interval and $J = (c, d) = (I + x)y$ then I is distributive if and only if Iy is nontrivial and $J = Iy + c$. (Equivalently $J = Id + c$).

***Proof*:** Under the hypothesis of the lemma let $I = (a, b)$ be distributive then $J = Ip + q$ for some p, q in L, then by Lemma 2.4 IJ cannot be trivial.

$$
\begin{aligned}
\text{Now } IJ &= (a\ (a+x)y,\ b(b+x)y) \\
&= (ay,\ by) \\
&= Iy.
\end{aligned}
$$

Hence Iy is nontrivial.

Conversely let Iy be nontrivial, then as the lattice L is modular and Iy is a lattice translate of a prime interval, it can only be a prime interval as shown in figure 2.1.

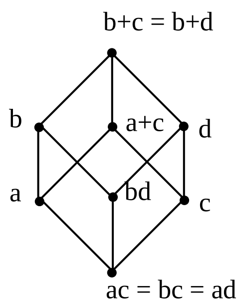


Figure 2.1

$$
\begin{aligned}
Iy &= (ay,\ by) \\
&= (a(a+x)y,\ b(b+x)y) \\
&= (ac,\ ad).
\end{aligned}
$$

Hence ac is covered by bd (Iy is a prime interval). Now L is a modular lattice of finite length hence we can define a dimension function d which satisfies $d(a+b) + d(ab) = d(a) + d(b)$. Thus we have

$$
\begin{aligned}
d(a+c) + d(ac) &= d(a) + d(c) \\
d(b+d) + d(bd) &= d(b) + d(d).
\end{aligned}
$$

Subtracting the former from the latter we get

$$
\begin{aligned}
&[d(b+d) - d(a+c) + [d(bd) - d(ac)] \\
&= [d(b) - d(a)] + [d(d) - d(c)]
\end{aligned}
$$

= 1 + 1 (as b covers a and d covers c).
= 2.

Now d(bd) – d(ac) = 1, implies d(b+d) – d(a+c) = 1 that is b + d covers a + c. Now a + c ≤ b + c ≤ c ≤ b + d and a + c is covered by b + d.

Either b + c = a + c or b + c = b + d.

If a + c = b + c then a + c > b

$\Rightarrow$ a + (a + x)y > b
$\Rightarrow$ (a + x) (a + y) > b
$\Rightarrow$ (a + x) > (a + x) (a + y) > b
$\Rightarrow$ a + x > b + x
$\Rightarrow$ a + x = b + x
$\Rightarrow$ (a + x)y = (b +x)y
$\Rightarrow$ J is trivial. A contradiction. Thus b + c = b + d.

Now ac ≤ bc ≤ bd and ac is covered by bd. So bc = ac or bc = bd. bc ≠ bd. For if b + c = b + d, as c < d and the lattice is modular, we have bc = ac.

Next ad = a (b + x)y = ay = ac. That is ad = ac.

Now,

abd = bad = bac = bbc = bc = ac
ac = ad = abd = bc

that is abd = bcd.

So ac = bcd which can be written as a c c = b d c. Now ac ≤ bd. So ac + c ≠ bd + c (L is modular) that is c ≠ c + bd. So c ≤ bd + c ≤ d. Also c is covered by d so bd + c = d that is J = Id + c = Iy + c.

Dually we have

Lemma 2.6: If I = (a, c) is a prime interval and J = (c, d) = Ix + y then J is a distributive lattice translate of I if and only if I + y is nontrivial and J = (I + y)d. Equivalently J = (I + c)d.

Lemma 2.7: If a is covered by b and c is covered by d with $b \leq c$ then $(c, d) \neq ((a + x)y, (b + x)y)$ for any $x, y \in L$.

***Proof*:** If $(c, d) = ((a + x)y, (b + x)y)$ and a is covered by b and $b \leq c$ and c is covered by d then $c = (a + x)y$.

$$\begin{aligned} c + x &= (a + x)y + x \\ &= (a + x)(x + y) \\ (c+x)y &= (a + x)y = c. \end{aligned}$$

But $(b + x)y$ lies between $(a + x)y$ and $(c+x)y$ as $a < b \leq c$, so equals c, a contradiction, a $c \neq d$ and $(b + x)y = d$.

Lemma 2.8: Let a is covered by b and c is covered by d with $d \leq a$ then $(c, d) \neq ((a + x)y, (b + x)y)$ for any x, y in L.

***Proof*:** If $d = (b + x)y$ then $d + x = (b + x)y + x$
$= (b + x)(x + y)$ implies
$(d + x)y = (b + x)y = d$.
Thus $d = (d + x)y < (a + x)y < (b + x)y = d$ implies
$(a + x)y = d = c$; a contradiction.

Corollary 2.1: If there exists intervals $I = (a, b)$, $J = (c, d)$ with $a < b \leq c < d$ then neither can be a distributive translate of the other in any general lattice.

Corollary 2.2: If $a < b \leq c < d$ in a distributive lattice then neither $J = (a, b)$ nor $J = (c, d)$ can be a lattice translate of the other.

DEFINITION 2.3: *A prime interval $J = (c, d)$ of a modular lattice L is called a distributive lattice translate of a prime interval $I = (a, b)$ if and only if J can be expressed as $(I + c)d$ or equivalently J can be expressed as $Id + c$.*

Lemma 2.9: If $(I + x)y = J$ with $IJ = Iy$ trivial then $I + J$ contains a five element modular sublattice and is of length 2.

***Proof*:** Length $(I + J)$ + Length $(IJ) = 2$. As Length $IJ = 0$ (given); Length $(I + J) = 2$.

Let $I = (a, b)$ and $J = (c, d)$, $IJ = 0$ implies $ac = bd$. Also $ac \leq bc \leq bd$ and $ac \leq ad \leq bd$. So $ac = bc$ and $ad = bd$. Now L is modular. Therefore $a + c \neq b + c$ and $a + d \neq b + d$. Also $ac = ad$ and $bc = bd$. Again as L is modular $a + c \neq a + d$ and $b + c \neq b + d$.

Next $b + c \neq a + d$. For if $b + c = a + d$ then $b + c + a + d = a + d$ then is $b + d = a + d$; a contradiction.

But $a + c$ is covered by $b + c$ and $b + c$ is covered by $b + d$ and $a + c$ is covered by $a + d$ and $a + d$ is covered by $b + d$.

Also $a + c = (a + x)(a + y)$ and $b + c = (b + x)(b + y)$. Let $p = (a + x)(b + y)$ as $b + d = (b + x)(b + y)$ either $(p, b + d)$ is trivial or p is covered by $b + d$. If $p = b + d$, then $(a + x)(b + y) = (b + x)(b + y)$. So $(a + x)(b + y)y = (b + x)(b + y)y$ will imply $(a + x)y = (b + x)y$ that is $c = d$; a contradiction.

So $(p, b + d)$ is nontrivial. Thus p is covered by $b + d$. Now $b + d$ covers $b + c$ and $a + d$. We assert neither $b + c = p$ nor $a + d = p$. For if $b + c = p$ then $(b + c, b + d) = (I + x)(b + d)$ and a is covered by $b \leq b + c$ and $b + c$ is covered by $b + d$; a contradiction in view of Lemma 2.7.

If $a + d = p$ then $(a + d, b + d) = [J + a + x)](b + d)$ and c is covered by $d \leq a + d$ and $a + d$ is covered by $b + d$; a contradiction in view of Lemma 2.8.

Thus $a + c$, $b + c$, p, $a + d$, $b + d$ is isomorphic to the five element modular lattice of length 2 contained in $I + J$ (cf. figure 2.2)

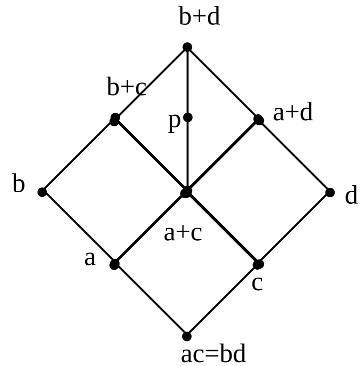


Figure 2.2

Dually we have

Lemma 2.10: If Ix + y = J with I + J = I + y trivial then IJ contains a five element modular sublattice and is of length 2.

Lemma 2.11: Let L be a modular lattice of finite length. Let $C:0 = x_0 < x_1 < x_2 < \ldots < x_n = 1$ be a maximal chain connecting 0 and 1. Let I = (a, b) (a is covered by b) be an arbitrary prime interval of L then I is a distributive lattice translate of a unique prime interval of C.

Proof: Now I = ((0, 1) + a) b as $(x_0 + a)b = a$ and $(x_n + a)b = b$. Next $a \leq (x_i + a)\, b \leq b$ for all x_i. Given a is covered by b; either $(x_1 + a)\, b = a$ or b; for all x_i.

Let k be the largest i for which $(x_k + a)b = a$; then $(x_{k+1} + a)b = b$.

So $I = ((x_k, x_{k+1}) + a)b$.

Now as L is modular and a is covered by b we have $I = ((x_k, x_{k+1})\, b) + a$ also.

Thus I = (a, b) is a distributive lattice translate of (x_k, x_{k+1}).

Further this (x_k, x_{k+1}) is unique. For if I is a distributive lattice translate of another (x_j, x_{j+1}) of C then

$I = (a, b) = ((x_j, x_{j+1}) + a)\, b$ by Lemmas 2.5 and 2.6.

Now as k and j are comparable without loss in generality we can assume $k > j$, then we see that $(x_j + a)\, b = (x_{j+1} + a)b = a$; a contradiction. Thus $k = j$ and so the uniqueness of the interval is established.

Remark: The modularity of the lattice is a necessary condition in Lemma 2.11.

***Proof*:** Consider the lattice of figure 2.3

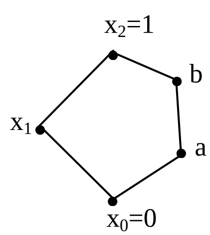


Figure 2.3

and the chain $C : 0 = x_0 < x_1 < x_2 = 1$. Let $I = (a, b)$ be the prime interval. Now I is not a distributive lattice translate of any interval in C.

For $((x_0, x_1) + a)b \quad = \quad (a, b) \neq (x_0, x_1)b + a$

$((x_1, x_2)b)+a \quad\quad = (a, b) \neq [(x_1, x_2) + a]b.$

Let L be a modular lattice of length n and $C : 0 = x_0 < x_1 < \ldots < x_n = 1$ be a maximal chain connecting 0 to 1. Let P denote the totality of all prime intervals of L. Partition P with respect to C into n classes $P_1, P_2, \ldots, P_n$ by the following procedure.

P_i = the set of all intervals P which are distributive lattice translates of (x_{i-1}, x_i).

Now it follows as a consequence of Lemma 2.11 above that any prime interval of L belongs to one and only one class P_i.

Lemma 2.12: If L is a distributive lattice then for each interval $J_i = (x_{i-1}, x_i)$ of C, θ_{J_i} annuls just those prime intervals of L which belong to P_i and no more.

***Proof*:** Follows as any lattice translate of any prime interval J_i of L is a distributive lattice translate of J_i.

Lemma 2.13: If J_i is a distributive interval of a modular lattice L then θ_{J_i} annuls just those prime intervals of L which belong to P_i and no more.

***Proof*:** Follows as in the case of Lemma 2.12.

Lemma 2.14: If a prime interval I = (a, b) of L is a distributive lattice translate of some interval $J_i = (x_{i-1}, x_i)$ of C then I is a lattice translate of J_i considered as intervals in the distributive sublattice generated by I and C.

Proof: As in this case

$$\begin{aligned} I &= ((x_{i-1}, x_i) + a)b \\ &= ((x_{i-1}, x_i)b) + a \end{aligned}$$

and all the elements involved belong to the sublattice generated by I and C.

Lemma 2.15: If I = (a, b), $I_1 = (a_1, b_1)$ are prime intervals of L such that a is covered by b and $b \leq a_1$ is covered by b_1 then the classes to which I and I_1 belong in the partition of P with respect to any arbitrary chain C are distinct.

***Proof*:** Let D denote the chain of L consisting of the intervals I and I_1; and let S be the distributive sublattice generated by C and D. Let P_1, P_2, …, P_n be the classes corresponding to the partitioning of the prime intervals of L with respect to the chain C; and $P_1', P_2', \ldots, P_n'$ be the classes corresponding to the partitioning of the prime intervals of S with respect to the chain C.

In view of Lemma 2.14 observe that the interval I belongs to P_i (i.e., the class containing (x_{i-1}, x_i)) if and only if I belongs to P_i' (i.e., the class (x_{i-1}, x_i)).

Now as S is a distributive lattice, I and I_1 cannot be lattice translates of each other in S; hence will belong to different classes under the partitioning of S with respect to C which in turn gives the required result.

Lemma 2.16: If C_1 is any other maximal chain connecting 0 to 1 of L then C_1 has exactly n prime intervals each of them belong to the classes P_1, P_2, …, P_n, taken in some order.

***Proof*:** Follows from Lemma 2.15.

Lemma 2.17: If a lattice translate $K = (J_i + x)y$ or $(J_i x + y)$ of J_i is non-distributive then K belongs to a class P_j different from P_i. Further the prime intervals of the five element sublattice $K + J_i$ (KJ_i) belong either P_i or to P_j.

***Proof*:** Lemma 2.9 and Lemma 2.10 assert the existence of the five element sublattice $K + J_i$ (and KJ_i) respectively consider any maximal chain C_1 of L connecting 0 and 1 which passes through the end points of $K + J_i$ (or KJ_i). From the previous lemma it follows that the prime intervals of C except those within $K+J_i$ (or KJ_i) belong to (n–2) classes of the partitioning of L with respect to C. Let the two classes which are omitted be P_i and P_j. These are the classes to which the prime intervals in $K + J_i$ (or KJ_i) belong irrespective of the element of $K + J_i$ (or KJ_i) occurring in the chain C_1.

Lemma 2.18: If the lattice translates of (x_{i-1}, x_i) and (x_{j-1}, x_j) of the previous lemma are contained completely within the classes P_i and P_j then $\theta_{(x_{i-1}, x_i)}$ annuals just the prime intervals belonging to these two classes.

***Proof*:** Follows as $\theta_{(x_{i-1}, x_i)}$ being the smallest congruence annulling (x_{i-1}, x_i) annuls just those intervals which can be written as a finite sum of lattice translates of (x_{i-1}, x_i).

Let K_1 be a non-distributive lattice translate of (x_{i-1}, x_i) or (x_{j-1}, x_j) lying outside the classes P_i and P_j, then the classes P_i and P_u (or P_j and P_u) meet in a five element modular lattice (where P_u denotes the class to which K_1 belongs). Thus to pass from one of the classes P_i to another class P_k one has to pass through a five element modular lattice.

Hence we have

THEOREM 2.1: *A modular lattice L of finite length n is simple if and only if the partition of L with respect to some arbitrary chain C satisfies property (α).*

"Any two of the classes P_1, P_2, …, P_n in the partitioning can be linked to one another by a sequence such that any two consecutive classes of the sequence meet at a five element modular lattice".

***Proof*:** Let $C : 0 = y_0 < y_1 < \ldots < y_n = 1$ be the chain. If L is simple then every prime interval of L is a lattice translate of any other prime interval. Now the prime interval $J = (y_1, y_2)$ of L is a lattice translate of $I = (y_0, y_1)$. Let

$$J = (I + x_1)x_2) + x_3\ldots) + x_{2r}$$

be a representation of J as a lattice translate of I. Let this representation be one of those which cannot be further reduced; that is $(I + x_1)x_2$ is a nondistributive translate of I, $(I + x_1)x_2 + x_3$ is a nondistributive translate of

$$I + x_1 \ldots ((I + x_1)x_2 + \ldots + x_{2i-2}) + x_{2i-1}$$

is a nondistributive translate of

$$(I + x_1)\, x_2 + x_3 \ldots) + x_{2i-3} \ldots \text{etc.}$$

Let I, $I+x_1 \in P_{\alpha_1} = P_1$

$((1 + x_1)\, x_2, (1+x_1)\, x_2 + x_3 \in P_{\alpha_2}$
$((1 + x_1)\, x_2+x_3)x_4, ((I+x_1)x_2 + x_3)x_4 + x_5 \in P_{\alpha_3}$
…
$J \in P_{\alpha_r} = P_2$.

$I \in P_1 = P_{\alpha_1}, P_{\alpha_2}, \ldots, P_{\alpha_r} = P_2$; $J \in P_2$ is the sequence by which these are linked. Similarly any P_i and P_j would be linked.

Conversely if a modular lattice L satisfies property (α) then any two of the prime intervals of L are lattice translates of each other and hence L is simple.

Remark: Start with any class P_1. Now this class P_1 should be linked to some class P_{α_2} so we should pass through a five element modular lattice. If those two are not further linked we would set a congruence on L just annulling these two classes and hence L will not be simple. So in the class of a simple lattice, atleast one of P_1 or P_{α_2} should be linked to another class P_{α_3} and this will give another five element modular lattice etc. This process will continue until all the classes are exhausted and so, we would atleast have (n–1) such five element modular lattices existing in L.

Conversely if we have a modular lattice L containing (n–1) such five element modular lattice in such a way that these link any two of the classes P_i with respect to some chain C then L is simple.

Corollary 2.3: L is a modular lattice containing a maximal chain C such that any two of the classes of the partition of P with respect to C satisfy property (α) then the classes of the partition of P with respect to any other chain C_1 also satisfy property (α).

***Proof*:** This follows as the first condition implies the simplicity of the lattice and the second is obtained as the choice of the chain C in the previous theorem is arbitrary.

Corollary 2.4: If L is a simple modular lattice then every class P_i has atleast one direct link with some other class P_j.

Corollary 2.5: Let L is a simple modular lattice of length n with $n \geq 3$, then there exists atleast one class P_i which has direct links with two or more classes.

THEOREM 2.2: *A simple modular lattice of length $n \geq 3$ contains a sublattice isomorphic to the lattice of figure 2.4 ($M_{3,3}$) or figure 2.5.*

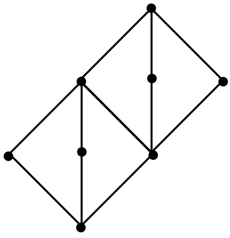


Figure 2.4 $M_{3,3}$

having a homomorphic image isomorphic to $M_{3, 3}$.

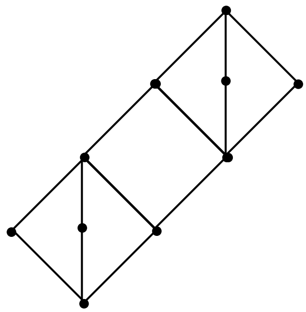


Figure 2.5 $M_{3,3}$

***Proof*:** Let P_k be the class mentioned in corollary 2.5 which has direct links with two or more classes then the two terms which the class P_k takes would give sublattices isomorphic to $M_{3,3}$ or to figure 2.5, that is a sublattice with a homomorphic image isomorphic to the lattice $M_{3,3}$.

Corollary 2.6: L is a simple modular lattice of length n, with n ≥ 4 then either there are (n–2) different sublattices of the type mentioned in the above lemma or there exists a sublattice in L isomorphic to the lattice of figure 2.6, figure 2.7, figure 2.8 or figure 2.9, that is a sublattice with a homomorphic image isomorphic to the lattice $M_{3,3,3}$.

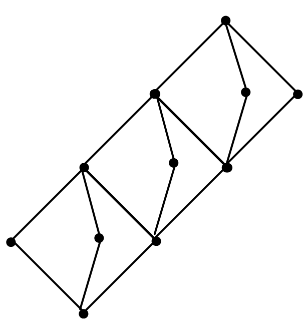


Figure 2.6 $M_{3,3,3}$

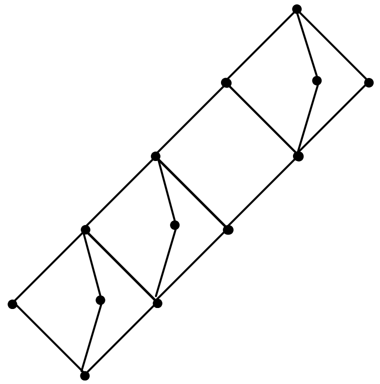


Figure 2.7 $\mathbf{M_{3,3,3}}$

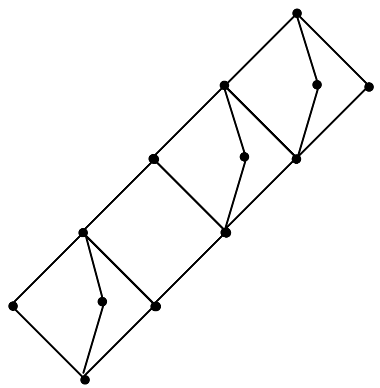


Figure 2.8 $\mathbf{M_{3,3,3}}$

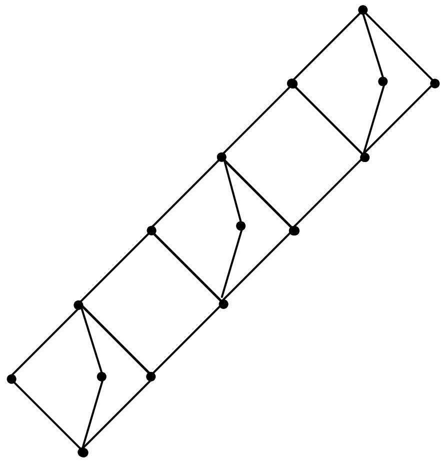


Figure 2.9 $\mathbf{M_{3,3,3}}$

Chapter Three

SUPERMODULAR LATTICES

Distributive lattices and modular lattices are the two well known equational classes of lattices. In this chapter we introduce another equational class of lattices - called the supermodular lattices. This equational class lies between the equational class of modular lattices and the equational class of distributive lattices.

It is well known that a modular lattice is nondistributive if and only if it contains a sublattice isomorphic to M_3. In a similar fashion, we prove that a modular lattice is nonsupermodular if and only if it contains a sublattice whose homomorphic image is isomorphic to M_4 or $M_{3,3}$.

Further we obtain (cf. Theorem 3.6). A super modular lattice is isomorphic to a subdirect union of copies of C_2 and M_3.

DEFINITION 3.1: *A lattice L is said to be supermodular if it satisfies the following identity*

$(a + b) (a + c) (a + d) = a + bc (a + d) + cd (a + b) + bd (a + c)$ *for all a, b, c, d in L.*

Lemma 3.1: Every supermodular lattice is modular.

***Proof*:** Put c = d

$$\begin{aligned}(a+b)(a+c) &= a + bc + c(a+b) + bc \\ &= a + c(a+b)\end{aligned}$$

is true for all a, b, c in L, which can easily be recognized as the modular law.

Lemma 3.2: Every modular lattice is not necessarily supermodular.

***Proof*:** By an example.

Consider the elements a, b, c, d as marked in the lattice M_4 of figure 3.1 then

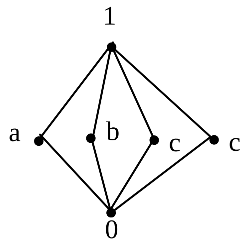


Figure 3.1 M_4

$$(a + b)(a + c)(a + d) = 1$$
$$a + bc(a + d) + cd(a + b) + db(a + c) = a.$$

Hence it is not supermodular.

Lemma 3.3: Every distributive lattice is supermodular.

***Proof*:** If L is distributive then

L.H.S. = R. H. S. = a + bcd.

Lemma 3.4: In a modular lattice L if a, b, c, d are 4 elements such that any two are comparable then this set of 4 elements satisfies the supermodular law.

***Proof*:** Let a > b.

$(a + b)(a + c)(a + d) = a \geq a + bc(a + d) + bd(a + c) + cd(a + b) \geq a.$

So the law holds.

Let a < b

L.H.S. $= b(a + c)(a + d) = b(a + c(a + d))$
$= a + bc(a + d).$

So $a + bc(a + d) \geq$ R.H.S. $\geq a + bc(a + d).$

So L.H.S. = R.H.S.

Let b > c

$(a + b)(a + c)(a + d) = (a + c)(a + d)$
$= a + c(a + d)$
$= a + bc(a + d).$

So

L.H.S. $= a + bc(a + d) \geq$ R.H.S.
$\geq a + bc(a + d).$

So the law is satisfied.

Lemma 3.5: Every supermodular lattice is not necessarily distributive.

***Proof*:** By an example.
The lattice of figure 3.2 is supermodular but not distributive.

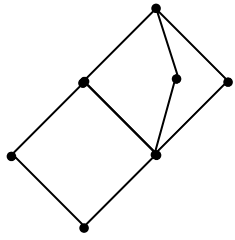


Figure 3.2

Supermodularity of L can easily be checked as L is a modular lattice and does not contain elements a, b, c, d such that any two of these are mutually incomparable (cf. Lemma 3.4).

Lemma 3.6: The lattices M_4 and $M_{3,3}$ are non-supermodular.

Proof: M_4 is already shown to be non supermodular in Lemma 3.2.

$M_{3,3}$ is not supermodular. For consider the elements a, b, c, d as shown in $M_{3,3}$ of figure 3.3 then $(a + b)(a + c)(a + d) = 1$ while $a + bc(a + d) + cd(a + b) + db(a + c) = a$.

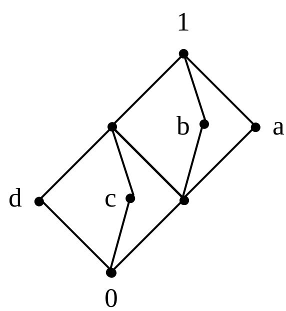


Figure 3.3 $M_{3,3}$

Lemma 3.7: Direct sum of supermodular lattices is supermodular.

Proof: Straight forward.

Lemma 3.8: Any sublattice of a supermodular lattice is supermodular.

Proof: Obvious.

Lemma 3.9: Any homorphic image of a supermodular lattice is supermodular.

Proof: Follows as the supermodular identity is a finite identity.

THEOREM 3.1: *The class of supermodular lattices is an equational class of lattices lying between the equational class of modular lattices and the equational class of distributive lattices.*

***Proof*:** Follows from Lemma 3.1, Lemma 3.3, Lemma 3.7, Lemma 3.8 and Lemma 3.9.

Lemma 3.10: If L is a supermodular lattice then no homomorphic image of L is isomorphic to M_4 and $M_{3,3}$.

Lemma 3.11: If L is a lattice which is not supermodular then L contains a set of 4 elements a, b_1, c_1, d_1 such that $(a + b_1)(a + c_1)(a + d_1) > a$ while $a > b_1 c_1 (a + d_1)$, $c_1 d_1 (a + b_1)$, $d_1 b_1 (a + c_1)$ holds.

(b_1, c_1, d_1) being distinct $b_1 \neq c_1$ otherwise $b_1 = b_1 c_1$ and $a + b_1 = a + b_1 c_1$, a contradiction as it will imply equality of $(a + b_1)(a + c_1)(a + d_1) = a$.

***Proof*:** Let L be a modular lattice which is not supermodular. As L is not supermodular there exist elements p, x, y, z such that

$$(p + x)(p + y)(p + z) \underset{\neq}{>} p + xy(p + z) + yz(p + x) + zx(p+y).$$

Put $a = p + xy(p + z) + yz(p + x) + xz(p + y)$

$b_1 = x$

$c_1 = y$

$d_1 = z$.

Then $a + b_1 = p + xy(p + z) + yz(p + x) + xz(p + y) + x$

$= p + x + yz(p + x)$

$= p + x$.

Similarly $a + c_1 = p + y$ and $a + d_1 = p + z$.

So $(a + b_1)(a + c_1)(a + d_1) \underset{\neq}{>} a$.

Lemma 3.12: If L is a modular lattice which is not supermodular then L contains a set of 4 distinct elements a, b, c, d such that

$$a + b = a + c = a + d \underset{\neq}{>} a.$$

Further $a > bc, cd, db$ holds.

***Proof*:** As L is not supermodular, by Lemma 3.11 we can assert the existence of a set of 4 elements a_1, b_1, c_1, d_1 in L such that

$$(a + b_1)(a + c_1)(a + d_1) \underset{\neq}{>} a$$

and

$$a > b_1c_1(a + d_1)$$
$$c_1d_1(a + d_1)$$

and

$$b_1d_1(a + c_1)$$

Put

$$b = b_1(a + c_1)(a + d_1)$$
$$c = c_1(a + b_1)(a + d_1)$$
$$d = d_1(a + b_1)(a + c_1)$$

then

$$a + b = a + b_1(a + c_1)(a + d_1)$$
$$= (a + b_1)(a + c_1)(a + d_1)$$

as $(a + c_1)(a + d_1) > a$ and L is modular.

Similarly

$$a + c = (a + b_1)(a + c_1)(a + d_1)$$

and

$$a + d = (a + b_1)(a + c_1)(a + d_1).$$

Therefore

$$a + b = a + c + a + d \underset{\neq}{>} a.$$

Now as

$$bc = b_1(a + c_1)(a + d_1)c_1(a + b_1)(a + d_1)$$
$$= b_1c_1(a + d_1).$$

We get $a > bc$. Similarly $a > cd$ and db.

Lemma 3.13: If L is a modular lattice which is non supermodular then for the set of 4 elements a, b, c, d of Lemma 3.12, all the three lattices generated by (a, b, c), (a, c, d) and (a, d, b) are non-distributive.

***Proof*:** Let if possible the lattice generated by (a, b, c) be a distributive lattice; then a + b = (a + b) (a + c) = a + bc = a (as a > bc); a contradiction. Similarly we can prove the nondistributive nature of the other two lattices.

Lemma 3.14: If L is a modular lattice which is non supermodular then for the set of 4 elements a, b, c, d of Lemma 3.13, the sublattices as shown in figures 3.4, 3.5, and 3.6 have homomorphic images isomorphic to the lattice $M_{3,3}$ of figure 3.3.

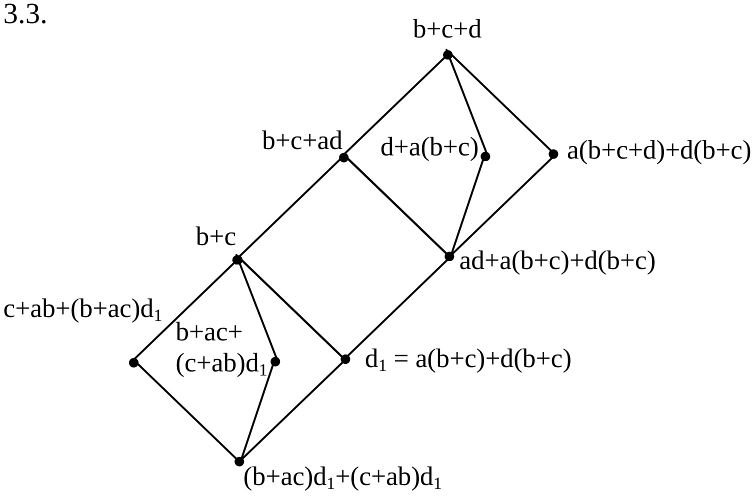


Figure 3.4

***Proof*:** As a + b = a + c = a + d we have

b + c + d < a + b

[a + c + ad] + [d + a(b + c)] = b + c + d

[b + c + ad] + [a (b + c + d) + d (b + c)]

= [[b + c + ad] + a (b + c + d)] + d (b + c)

= (b + c + d) [a + b + c + ad] + d (b + c) (L is modular)

= (b + c + d) + d (b + c) (as a + b > b + c + d)

= b + c + d.

[d + a (b + c)] + [a (b + c + d) + d (b + c)]

= d + a (b + c + d)

= (d + a) (b + c + d) (L is modular)

= b + c + d (as a + d > b + c + d).

So much for the unions for the first diamond. Next

$[b + c + ad]\ [d + a\ (b + c)]$

$= ad + (b + c)\ [d + a(b + c)]$

(L is modular and $ad < d + a\ (b + c)$).

$= ad + d\ (b + c) + a\ (b + c)$

(L is modular and $b + c > a\ (b + c)$).

$[b + c + ad]\ [a\ (b + c + d) + d\ (b + c)]$

$= d\ (b + c) + [b + c + ad]\ [a\ (b + c + d)]$

(L is modular, $d\ (b + c) < b + c + ad$)

$= d\ \ (b + c) + (b + c + d)\ (a\ (b + c) + ad)$

(L is modular and $a > ad$)

$= d\ (b + c) + a\ (b + c) + ad$

(as $(b + c + d) > a\ (b + c) + ad$).

$[d + a\ (b + c)]\ [a\ (b + c + d) + d\ (b + c)]$

$= d\ (b + c) + [d + a\ (b + c)]\ [a\ (b + c + d)]$

(L is modular and $d(b+c)$, $d + a(b+c)$

$= d\ (b+c) + [ad + a\ (b+c)]\ [b + c + d]$

(L is modular and $a > a\ (b + c)$)

$= d\ (b + c) + ad + a\ (b + c)$ (as $ad + a\ (b + c) < (b + c + d)$).

Thus the vertification for the upper diamond is complete.

As for the middle square

$(b + c) + [ad + a\ (b + c) + d(b + c)]$

$= b + c + ad.$

$(b + c)\ [ad + a\ (b + c) + d\ (b + c)]$

$= a\ (b + c) + d\ (b + c) + ad\ (b + c)$

(as L modular and $(b + c) > a\ (b + c) + d\ (b + c)$)

$= a\ (b + c) + d\ (b + c).$

Next we come to the lower diamond.

$[(b + ac) + (c + ab)d_1] + d_1$

$= (b + ac) + d_1$

$= (b + ac) + a\ (b + c) + d\ (b + c)$

$= (b + c)\ (a + b + ac) + d\ (b + c)$

(L is modular and $(b + c) > b + ac$).

$= (b + c) + d\ (b + c)$

[as $a + b > b + c$ by Lemma 3.12]

$= b + c.$

$[(c + ab) + (b + ac)d_1] + d_1$
$= (c + ab) + d_1$
$= (c + ab) + a\,(b + c) + d\,(b + c)$
$= (b + c)\,(a + c + ab) + d\,(b + c)$
(L is modular and $(b + c) > c + ab]$
$= b + c + d(b + c)$
(as $a + c > b + c$ by Lemma 3.12)
$= b + c$
$[(b + ac) + (c + ab)\,d_1] + [(c + ab) + (b + ac)d_1]$
$= b + c.$

Next for the intersections in the lower diamond we have
$d_1\,[(b + ac) + (c + ac)d_1]$
$= (b + ac)d_1 + (c + ab)d_1$ (L is modular)
$d_1\,[(c + ab) + (b + ac)d_1]$
$= (c + ab)d_1 + (b + ac)d_1$ (L is modular).
$[(b + ac) + (c + ab)d_1]\quad [(c + ab) + (b + ac)d_1]$
$= (c + ab)d_1 + (b + ac)\,[(c + ab) + (b + ac)d_1]$
[L is modular and $(c + ab)d_1 < c + ab + (b + ac)d_1]$
$= (c + ab)d_1 + (b + ac)d_1 + b(a + ac)\,(c + ab)$
[L is modular and $(b + ac)d_1 < (b + ac)]$
$= (c + ab)d_1 + (b + ac)d_1 + ac + b\,(c + ab)$
[L is modular and $ac < c + ab]$
$= (c + ab)d_1 + (b + ac)d_1 + ac + bc + ab$
[L is modular and $b > ab]$
$= (c + ab)d_1 + (b + ac)d_1 + ab + ac$
[as $a > bc$ implies $ab > bc]$
$= (c + ab)d_1 + (b + ac)d_1$
$[(b + ac)d_1 > ac,\ (c + ab)d_1 > ab].$

Thus the proof is complete for figure 3.4. The proof in the case of figures 3.5 and 3.6 follows from symmetry in the figures and as there exists perfect symmetry in the elements b, c, d. Now the homomorphism which annuls $(b + c,\ b + c + ad)$, $(d_1,\ d_1 + ad)$ gives the homomorphic image isomorphic to the lattice $M_{3,3}$ of figure 3.3.

Lemma 3.15: If L is a modular lattice which is non supermodular then for the four elements a, b, c, d of Lemma

3.14 in the case b + c + d = b + c + ad = c + d + ab = d + b + ac; there exists either a sublattice isomorphic to the lattice M_4 as shown in figure 3.7.

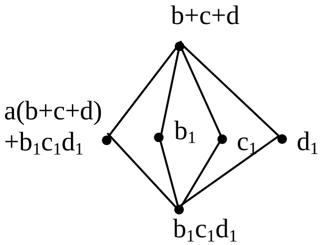


Figure 3.7

or a sublattice isomorphic to $M_{3,3}$ as shown in figures 3.8 and 3.9.

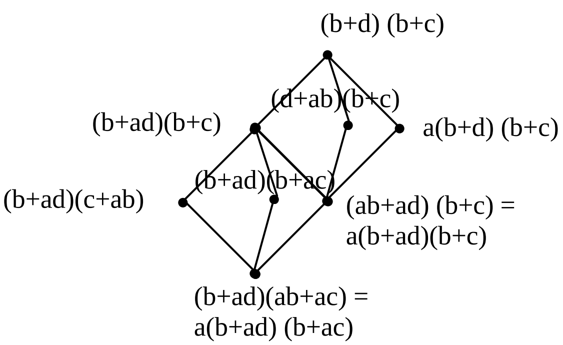


Figure 3.8

***Proof*:** Given b + c + ad = b + c + d = c + d + ab = d + b + ac.

Consider
b_1 = b + ac + ad
c_1 = c + ab + ad
d_1 = d + ab + ac

then $b_1 + c_1 = c_1 + d_1 = d_1 + b_1$ = b + c + d. Also a + b_1 = a + b = a + c_1 = a + d_1.

Let $a_1 = a (b_1 + c_1)$

then

$a_1 + b_1 = a (b_1 + c_1) + b_1$

$= (a + b_1) (b_1 + c_1)$

$= (b_1 + c_1)$ [as $a + b_1 = a + b > b_1 + c_1$].

Similarly

$a_1 + c_1 = b_1 + c_1$

$a_1 + d_1 = a (c_1 + d_1) + d_1$

$= (a + d_1) (c_1 + d_1)$

$= c_1 + d_1$

$= b_1 + c_1$.

Hence the sum of any two of a_1, b_1, c_1, d_1 equals $b + c + d$.

Next

$a_1b_1 = a (b_1 + c_1)b_1$

$= ab_1$

$= a (b + bc + ad)$

$= ab + ac + ad$.

Similarly,

$a_1c_1 = a_1d_1 = ab + ac + ad$

$b_1c_1 = (b + ac + ad) (c + ab + ad)$

$= ac + ad + b (c + ab + ad)$

(L is modular and $ac + ad < (c + ab + ad)$)

$= ab + ac + ad + b (c + ad)$.

Therefore

$b_1c_1 \geq ab + ac + ad$.

Similarly

$c_1d_1 > ab + ac + ad$

and

$d_1b_1 > ab + ac + ad$.

Now two cases arise

(1) Either $b_1c_1 = c_1d_1 = d_1b_1 = b_1c_1d_1$ or one of $b_1c_1 \neq b_1c_1d_1$. In the first case we have the lattice as shown in figure 3.7 isomorphic to the lattice M_4 consisting of $b + c + d$, $a_2 = (a_1 + b_1c_1d_1)$, b_1, c_1, d_1 and $b_1c_1d_1$.

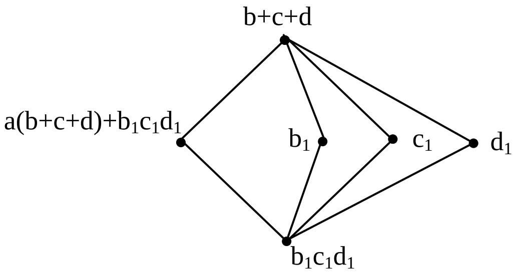


Figure 3.7

If further the elements b + ac + ad, c + ab + ad, ab + ac + d, b + c + d coincide, then the sublattice isomorphic to M_4 collapses.

In this situation the elements (b + d) (b + c), a (b+d) (b+c), (d+ab) (b+c), (b + ad) (b+c), (ab+ad) (b+c), (b+ad) (b+ac), (b+ad) (c+ab) (c+ab) and (b+ad) (ab+ac) form a sublattice isomorphic to $M_{3,3}$ of the lattice as shown in figure 3.8.

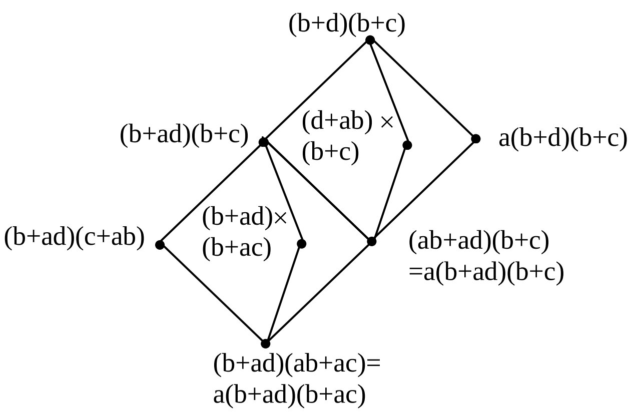


Figure 3.8

For a (b+d) (b+c) + (d+ab) (b+c)

= (b+d) (b + c) [a(b+c) + d + ab]

(b+c+d $\geq$ a (b+c) + d + ab

$\geq$ ab + ac + d = b + c + d)

= (b + d) (b + c) (b + c + d)

= (b + d) (b + c).

$a(b+d)\ (b+c) + (b+ad)\ (b+c)$
$= (b+c)\ (b+d)\ (a(b+c) + b + ad)$
$= (b+c)\ (b+d)\ (b+c+d)$
$= (b+c)\ (b+d).$

$(b+c)\ (d+ab) + (b+c)\ (b+ad)$
$= (b+c)\ (d+ab+(b+c)\ (b+ad))$
(as $ab < (b+c)\ (b+ad)$)
$= (b+c)\ (d+b+c\ (b+ad))$
(as $c(b+ad) < (b+ad)\ b+d$)
$= (b+c)\ (b+d).$
$(b+c)\ (b+ad)\ a(b+c)\ (b+d)$
$= (b+c)\ a(b+ad) = (b+c)\ (ab+ad).$

Next $(b+c)\ (d+ab)\ a(b+c)\ (b+d) = (b+c)\ a(d+ab)$
$= (b+c)\ (ab+ad).$

Also $(b+c)\ (b+ad)\ (b+c)\ (d+ab)$
$= (b+c)\ (ab+(b+ad)d)$
$= (b+c)\ (ab+bd+ad)$
$= (b+c)\ (ab+bd+ad)$ (as $a > bd$ so $ad > bad$)
$= (b+c)\ (ab+ad).$

Now for the lower diamond
$(b+c)\ (ab+ad) + (b+ac)\ (b+ad)$
$= ab + ad\ (b+c) + b + ad(b+ac)$
$= b + ad(b+c) = (b+ad)\ (b+c).$

$(b+c)\ (ab+ad) + (b+ad)\ (c+ab)$
$= ab + ad\ (b+c) + ab + c(b+ad)$
$= ab + ad\ (b+c) + ab + c\ (b+ad)$
$= ab + (b+ad)\ (ad\ (b+c) + c)$
$= ab + (b+ad)\ (ad + c)\ (b+c)$
$= (b + ad)\ (b + c)\ (ab + ad + c)$
$= (b + ad)\ (b+c)\ (b+c+d)$
(as $c + ab + ad = b + c + d$)
$= (b + ad)\ (b + c).$

$(b+ac)\ (b + ad) + (b+ad)\ (c+ab)$
$= b + ad\ (b+ac) + ab + c(b+ad)$
$= b + (b+ad)\ (c+ad\ (b+ac))$
$= (b+ad)\ (b+c).$

As for intersections

$(b+c)\ (ab+ad)\ (b+ac)\ (b+ad)$
$= (b+ac)\ (ab+ad) = a\ (b+ad)\ (b+ac)$
$(b+c)\ (ab+ad)\ (b+ad)\ (c+ab)$
$= a\ (b+ad)\ (c+ab) = a(b+ad)\ (b+ac).$

$(b+ad)\ (b+ac)\ (b+ad)\ (c+ab)$
$= (b+ad)\ (b+ac)\ (c+ab)$
$= (b+ad)\ (ab+c(b+ac))$
$= (b+ad)\ (ab+cb+ac)$
$= (b+ad)\ (ab+ac)$ (as $a > bc$, $ab > bc$)
$= a\ (b+ad)\ (b+ac).$

Further $M_{3,3}$ cannot collapse, as this will mean
$a(b+d)\ (b+c) = (b+d)\ (b+c)$
implies
$a(b+d)\ (b+c) + a(b+d) = (b+d)\ (b+c) + a(b+d)$
that is
$a(b+d) = (b+d)\ (b+c+a(b+d))$
$= (b+d)\ (b+c+d)$
as $b+c+d = b+c+ad < b+c+a\ (b+d) < (b+c+d)$
that is $a(b+d) = (b+d)$;

a contradiction as (a, b, d) generates a non-distributive lattice, which means the sublattice of the figure 3.9 exists.

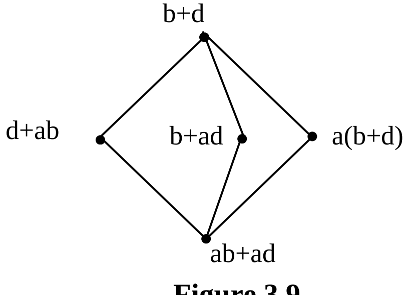


Figure 3.9

(2) Next let us consider the alternate possibility. That is among b_1c_1, c_1d_1, d_1b_1 atleast one is different from $b_1c_1d_1$. Without loss in generality let it be b_1c_1, that is $b_1c_1 > b_1c_1d_1$.

Then the elements (b_1+d_2) (c_1+d_2) $c_1(b_1+d_2)$, $b_1(c_1+d_2)$, $(a_1+b_1c_1)$ $(d_1+b_1c_1) = d_2$, $(a_1+b_1c_1d_1)$ $(d_1+b_1c_1)$, d_1 $(a_1+b_1c_1)$, b_1c_1 and $b_1c_1d_1$ form a sublattice isomorphic to the lattice of figure 3.10 For

$$b_1c_1 [(a_1 + b_1c_1d_1) (d_1+b_1c_1)]$$
$$= b_1c_1 (a_1 + b_1c_1d_1)$$
$$= a_1b_1c_1 + b_1c_1d_1 \text{ (L is modular)}$$
$$= b_1c_1d_1 \text{ (as } b_1c_1d_1 > a_1b_1c_1).$$

Also $b_1c_1 [d_1 (a_1+b_1c_1)] = b_1c_1d_1$.
Next

$$[d_1 (a_1+b_1c_1)] [(a_1+b_1c_1d_1) (d_1+b_1c_1)]$$
$$= d_1 (a_1+b_1c_1d_1)$$
$$= a_1d_1 + b_1c_1d_1 \text{ (L is modular)}$$
$$= b_1c_1d_1 \quad \text{(as } a_1d_1 < b_1c_1d_1).$$

Now for unions for the lower diamond

$$b_1c_1 + [(d_1 + b_1c_1) (a_1+b_1c_1d_1)]$$
$$= (d_1 + b_1c_1) [b_1c_1 + a_1 + b_1c_1d_1] \quad \text{(L is modular)}$$
$$= (d_1 + b_1c_1) (a_1 + b_1c_1)$$
$$= d_2.$$

$$b_1c_1 + [d_1 (a_1 + b_1c_1)]$$
$$= (b_1c_1 + d_1) (a_1 + b_1c_1) \quad \text{(L is modular)}$$
$$= d_2.$$

$$[(a_1 + b_1c_1d_1) (d_1 + b_1c_1)] + [d_1 (a_1 + b_1c_1)]$$
$$= (d_1 + b_1c_1) [a_1 + b_1c_1d_1 + d_2 (a_1 + b_1c_1)]$$
$$\text{(L is modular)}$$
$$= (d_1 + b_1c_1) [a_1 + d_1 (a_1 + b_1c_1)]$$
$$= (d_1 + b_1c_1) [(a_1 + d_1) (a_1 + b_1c_1)]$$
$$= (d_1 + b_1c_1) (a_1 + b_1c_1) \quad \text{(as } a_1+d_1 = a_1 + b_1).$$

For the intersections in the upper diamond

$[b_1 (c_1+d_2)]\ [c_1 (b_1+d_2)] = b_1c_1$

$[b_1 (c_1 + d_2)]\ d_2 = b_1d_2$

$$= b_1 (a_1 + b_1c_1)(d_1 + b_1c_1)$$
$$= (a_1b_1 + b_1c_1)(d_1 + b_1c_1)$$
$$= b_1c_1 (d_1 + b_1c_1) \quad (\text{as } a_1b_1 < b_1c_1)$$
$$= b_1c_1.$$

Similarly

$c_1d_2 = b_1c_1.$

As regard unions

$b_1 (c_1+d_2) + d_2 = (b_1 + d_2)(c_1 + d_2).$

Similarly

$c_1 (b_1 + d_2) + d_2 = (c_1 + d_2)(b_1 + d_2).$

Also

$b_1 (c_1+d_2) + c_1 (b_1 + d_2)$
$= (b_1 + d_2) [b_1 (c_1 + d_2) + c_1]$
$= (b_1 + d_2) [(b_1 + c_1)(c_1 + d_2)]$
$= (b_1 + d_2)(c_1 + d_2) \quad (\text{as } b + c_1 > b_1 + d_2).$

Thus the sublattice of figure 3.10

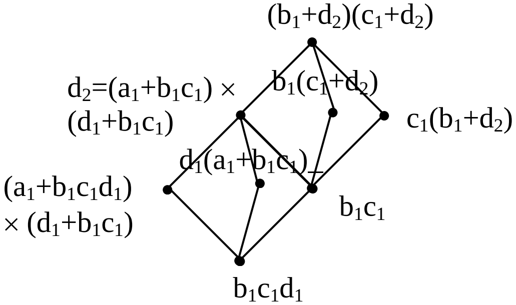


Figure 3.10

exist. No two elements of this lattice can equal as this will imply the equality of b_1c_1 and $b_1c_1d_1$ - a contradiction. Thus the proof of Lemma 3.15 is complete.

Combining these lemmas we obtain.

THEOREM 3.2: *If L is a modular lattice having a set of 4 elements a, b, c, d satisfying the following*

(1) $a + b = a + c = a + d \underset{\neq}{>} a$

(2) $a > bc, cd, db$.

Then either L has a sublattice isomorphic to M_4 or contains a sublattice whose homomorphic image is isomorphic to $M_{3,3}$.

Theorem 3.2 with Lemma 3.12 gives.

THEOREM 3.3: *A modular lattice L is supermodular if and only if L has no sublattice whose homomorphic image is isomorphic to M_4 and $M_{3,3}$.*

Corollary 3.1: A lattice is supermodular if and only if L does not contain a pertagon or sublattice whose homomorphic images are isomorphic to M_4 and $M_{3,3}$.

Lemma 3.16: The dual of a lattice L is supermodular if and only if there exists a set of four elements a, b, c, d in L such that $ab = ac = ad < a$; and $a < b + c, c + d, d + b$.

***Proof*:** Easy verification

As the dual of a modular lattice is modular and the lattices M_4 and $M_{3,3}$ are self dual.

By duality, we obtain

THEOREM 3.4: *A modular lattice L is dually supermodular if and only if L has no sublattice isomorphic to M_4 and has no sublattice whose homomphic image is isomorphic to $M_{3,3}$.*

As an immediate corollary we get.

THEOREM 3.5 *A lattice L satisfies*

$(a + b)(a + c)(a + d) = a + bc(a + d) + cd(a + b) + db(a + c)$ for all a, b, c, d in L if and only if it satisfies $ab + ac$

+ ad = a (b + c + ad) (c + d + ab) (a + b + ac) for all a, b, c, d in L.

***Proof*:** The proof follows as any supermodular or dually supermodular lattice is modular.

Next we obtain a characterization of the subdirectly irreducible supermodular lattices using a result due to B. Johnson [1968] [cf. Theorem of preliminaries) combined with a result due to G. Gratzer (1966).

"A subdirectly irreducible modular lattice of length n ≥ 3 contains a sublattice whose homomorphic image is isomorphic to $M_{3,3}$.

THEOREM 3.6: *A subdirectly irreducible supermodular lattice is isomorphic to the two element chain C_2 or the five element modular lattice M_3.*

***Proof*:** Let L be a subdirectly irreducible lattice. If length of L is 1 then L is isomorphic to C_2. If length of L is 2 then as L is subdirectly irreducible it can either be M_3 or has a sublattice isomorphic to the lattice M_4.

But L is supermodular and hence the other possibilities cease to exist. Hence L is isomorphic to M_3.

If length of L ≥ 3 then as L is subdirectly irreducible and modular (being supermodular), we can apply B. Johnson's result [ ] and obtain the sublattice isomorphic to $M_{3,3}$ which in turn gives rise to a contradiction as then L ceases to be supermodular. Thus the proof of the theorem.

Combining this with the famous Birkhoff's theorem we obtain.

Theorem 3.7: *A lattice L is supermodular if and only if L is a subdirect union of two element chain and the five element modular lattice M_3.*

***Proof*:** If L is supermodular, then by Birkhoff's theorem it is a subdirect union of subdirectly irreducible supermodular lattices. Using the above we see that any subdirectly irreducible supermodular lattice is either C_2 or M_3.

Hence L is a subdirect union of copies of C_2 and M_3.

For the converse it will suffice to see that both C_2 and M_3 are supermodular. Also by lemma 3.7 and 3.8 direct union of supermodular lattices is supermodular and sublattice of a supermodular lattice is supermodular. Hence subdirect unon of supermodular lattices is supermodular. So a subdirect union of copies of C_2 and M_3 is supermodular.

Corollary 3.1: If a lattice L is supermodular then every element of L has atmost two relative complements in any interval.

***Proof*:** Follows from Theorem 3.7.

Note: The converse of Corollary 3.2 is not necessarily true.

***Proof*:** By an example.

The lattice $M_{3,3}$ satisfies the requirements that every element of L has atmost two relative complements in any interval but L is obviously not supermodular.

Corollary 3.2: Let L be a supermodular lattice and let I = (x_1, x_2) be a prime interval in L. Let $I_1 = (y_1, y_2)$ and $I_2 = (z_1, z_2)$ be any two intervals of L such that I, I_1 and I_2, belong to a chain of L. If I_1 and I_2 are annulled by θ_I (the congruence generated by I) then atleast one of I_1, I_2 is a trivial interval.

***Proof*:** It is a direct consequence of Theorem 3.6.

Remark 3: The primeness of the interval I = (x_1, y_1) is absolutely essential in the above corollary. Otherwise θ_I can be written as a nontrivial sum of two other congruence and hence I may have more than an interval as its lattice translate belonging to a chain of L.

(For example the interval I of the supermodular lattice L of the adjoining figure 3.11).

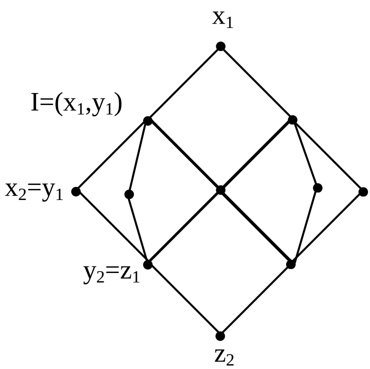


Figure 3.11

Remark 3.2: The converse of Corollary 3.3 is not necessarily true.

***Proof*:** By an example

Consider M_4 which satisfies the hypothesis for the converse of Corollary 3.3. Nevertheless it is not supermodular.

Corollary 3.4: The congruence generated by a prime interval is separable in any supermodular lattice.

***Proof*:** It is a direct consequence of Corollary 3.3.

Chapter Four

SEMI-SUPERMODULAR LATTICES

In the last chapter we studied supermodular lattices. An equational class of modular lattices generated by the finite modular lattice M_3. In this chapter, we initiate a study into a series of equational classes of modular lattices termed n-semi-supermodular lattices, for each finite integer n. The classes of distributive and supermodular lattices correspond to the integers n = 2,3 respectively. For $n \geq 4$, we observe that these equational classes are no longer generated by their finite members. Further we show that a lattice L is n-semi-supermodular if and only if it does not contain any sublattice whose homomorphic image is isomorphic to $M_{i_1+2, i_2+2, \ldots, i_k+2}$ and $\hat{M}_{i_1+2, i_2+2, \ldots, i_k+2}$ such that i_1, i_2, …, i_k are integers ≥ 1 with $i_1 + i_2 + \ldots + i_k = (n-1)$.

To avoid cumber some calculations we give rigorous proof in the case of n = 4 and indicate that the method of proof adopted for n = 4 can be extended for any general n (finite).

We start with the definition of 4-semi-supermodularity in lattices, which we choose to call semi-supermodular leaving the integer 4 for convenience.

DEFINITION 4.1: *A lattice L is said to be semi-supermodular if it satisfies the following identity.*

$(a+x_1)(a+x_2)(a+x_3)(a+x_4) = a + x_1x_2(a + x_3)(a + x_4) + x_1x_3(a + x_2)(a + x_4) + x_1x_4(a + x_2)(a+x_3) + x_2x_3(a+x_1)(a+x_4) + x_2x_4(a+x_1)(a+x_3) + x_3x_4(a+x_1)(a+x_2)$ for all a, x_1, x_2, x_3, x_4 in L.

Lemma 4.1 Any semi-supermodular lattice is modular.

***Proof*:** Put $x_2 = x_3 = x_4$ we get the modular law.

THEOREM 4.1: *A modular lattice L is not semi-supermodular if and only if L contains elements a, x_1, x_2, x_3, x_4 such that $a + x_1 = a+x_2 = a + x_3 = a + x_4 > a$ and $a > x_1x_2$, x_1x_3, x_1x_4, x_2x_3, x_2x_4 and x_3x_4.*

***Proof*:** If L contains elements a, x_1, x_2, x_3, x_4 as specified in the lemma then for this set of 5 elements of L the left hand side equals $a + x_1$ while the right hand side equals a and the two are distinct.

Hence L is not semi-supermodular.

Conversely if L is not semi-supermodular then there exist a set of 5 elements a, x_1, x_2, x_3, x_4 in L such that semi-supermodular identity is not satisfied in L.

Let

$x_1 = x_1(a + x_2)(a+x_3)(a+x_4)$

$x_2 = (a + x_1)x_2(a+x_3)(a+x_4)$

$x_3 = (a + x_1)(a + x_2)x_3(a+x_4)$

$x_4 = (a + x_1)(a+x_2)(a+x_3)x_4.$

A = R.H.S. of the semi-supermodular identity when a, x_1, x_2, x_3, x_4 are substituted in it. The set of 5 elements A, x_1, x_2, x_3 and x_4 satisfy the requirements of the lemma.

As a corollary we get.

Corollary 4.1: Any supermodular lattice is semi-supermodular.

***Proof*:** Equivalently we can prove if a lattice is not semi-supermodular then it is not supermodular. This follows as a consequence of the last lemma and lemma of Chapter 2 which states a lattice is not supermodular if and only if it contains a set of 4 elements a, x_1, x_2, x_3 such that $a + x_1 = a + x_2 = a + x_3 \geq a$ and $a > x_1x_2$, x_1x_3 and x_2x_3.

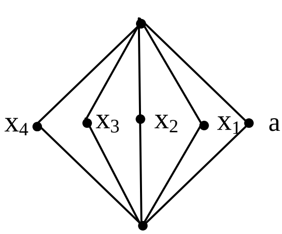
Lemma 4.2: Any semi-supermodular lattice is not necessarily supermodular.

***Proof*:** By examples.

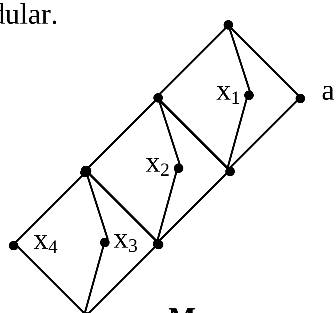
These lattices M_4 and $M_{3,3}$ are semi-supermodular but not supermodular.

Thus the equational class of semi-supermodular lattice lies between the equational class of modular lattices and the equational class of supermodular lattices.

Lemma 4.3: If a lattice L contains a sublattice whose homomorphic image is isomorphic to the lattices of the figures 4.1 then L is not semi-supermodular.



M_5



$M_{3,3,3}$

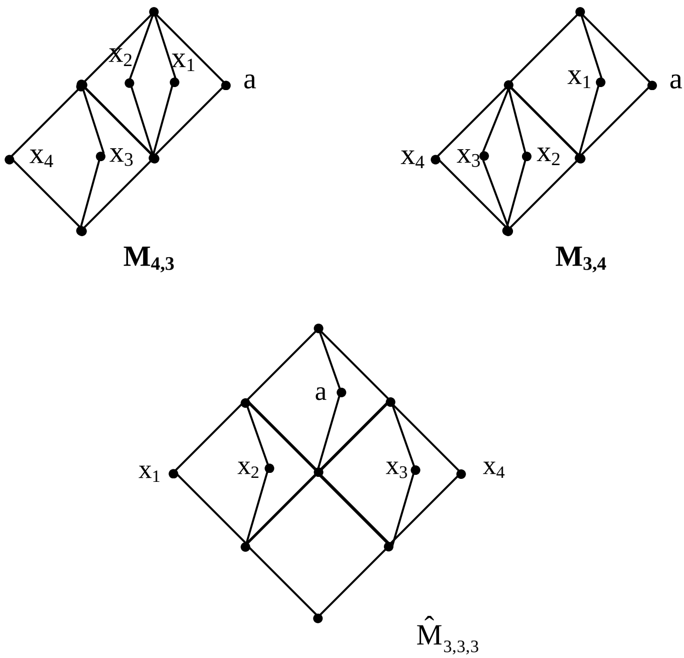


Figure 4.1

***Proof*:** The set of 5 elements as marked in the figures satisfy the requirements of theorem 4.1 and hence these cannot be got as homomorphic images of sublattices of semi-supermodular lattices.

Now let L be a modular, non semi-supermodular lattice, then by Theorem 4.1, L contains a set of 5 elements a, x_1, x_2, x_3, x_4 such that

$$a + x_1 = a + x_2 = a + x_3 = a + x_4 > a$$

with $a > x_i\, x_j$ ($i \neq j$ for all $i, j = 1, 2, 3, 4$)

Consider the set T of elements given by

$x_1 + x_2 + x_3 + ax_4$
$x_1 + x_2 + ax_3 + x_4$
$x_1 + ax_2 + x_3 + x_4$
$ax_1 + x_2 + x_3 + x_4$.

Two cases may arise;

I. At least one of the elements in T is different from $x_1 + x_2 + x_3 + x_4$.

II. Every member of T equals $x_1 + x_2 + x_3 + x_4$.

If case I occurs, let us without loss in generality assume that one of the elements different from $x_1 + x_2 + x_3 + x_4$ is $x_1 + x_2 + x_3 + ax_4$.

Now as $x_1 + x_2 + x_3 + ax_4 < x_1 + x_2 + x_3 + x_4$. We have a sublattice isomorphic to M_3 consisting of elements.

$x_1 + x_2 + x_3 + x_4$, $U = a(x_1 + x_2 + x_3 + x_4)$

$x_4 (x_1 + x_2 + x_3)$ $V = a(x_1 + x_2 + x_3) + x_4$

$W = x_1 + x_2 + x_3 + ax_4$ and

$X_1 = a(x_1 + x_2 + x_3) + ax_4 + x_4 (x_1 + x_2 + x_3)$ as shown in figure 4.2.

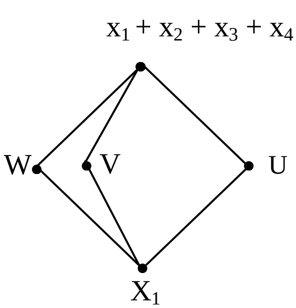


Figure 4.2

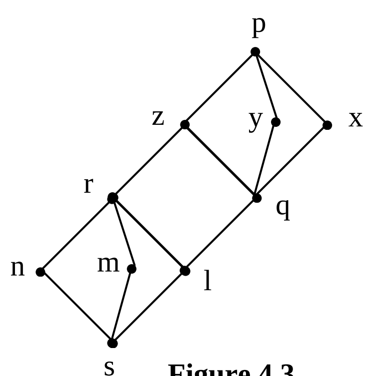


Figure 4.3

Now the set a, x_1, x_2, x_3 satisfies all the properties required for theorem 4.1. In case it has a sublattice isomorphic to $M_{3,3}$ of figures 3.4, 3.5, 3.6 of chapter III then let S be renamed as shown in the figures 4.3 then $p < x_1 + x_2 + x_3 + ax_4$ and not less than or equal to $a (x_1 + x_2 + x_3) + ax_4 + x_4 (x_1 + x_2 + x_3) = X_1$. Put $X_1 + x = K_1$ then K_i lies between X_1 and $x_1 + x_2 + x_3 + ax_4$ and $X = K_1p$ lies between x and p. Let $Y = y + X_z$ and $Z = z+X_y$ then p, X, Y, Z, $X_y + X_z$ is isomorphic to M_3. Also $l_1 = r (X_y + X_z)$ lies between l and r. Let $m_1 = l_1n + m$ and $n_1 = l_1m + n$ then

$r, l_1, m_1, n_1, l_1m + l_1m$ is isomorphic to M_3. Also the elements $(K_1 + U)(K_1 + V)$, $(K_1 + U)$, (K_1+V), $W + (K_1 + U)(K_1 + V)$, $(x_1 + x_2 + x_3 + x_4)$ form a sublattice isomorphic to M_3. These three sublattices M_3 combine to give us to required $M_{3,3,3}$ (cf. figure 4.4)

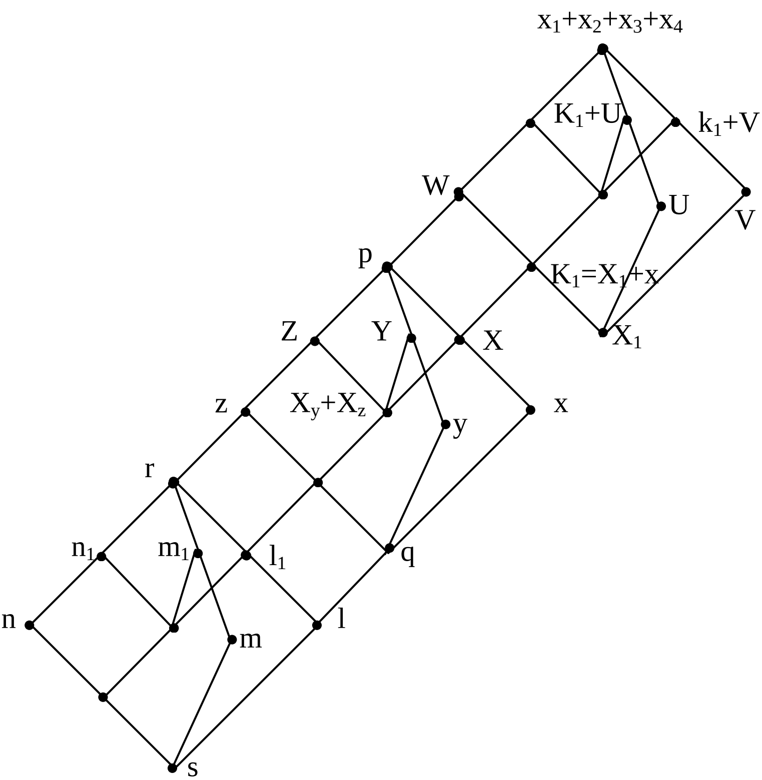


Figure 4.4

The only difficulty we may face is when $K_1 = x_1 + x_2 + x_3 + ax_4$. But this could be avoided by choosing $a > x_3 (x_1 + x_2)$ in which case K_1 will coincide with x_1; thus avoiding the difficulty.

Next if a, x_1, x_2, x_3 satisfy the condition $x_1 + x_2 + ax_3 = x_1 + ax_2 + x_3 = ax_1 + x_2 + x_3 = x_1 + x_2 + x_3$; then let $b_1 = x_1 + ax_2 +$

ax_3; $c_1 = ax_1 + x_2 + ax_3$; $d_1 = ax_1 + ax_2 + x_3$ and $a_1 = a(x_1 + x_2 + x_3)$.

In case (1) $b_1c_1 = c_1d_1 = d_1b_1 = b_1c_1d_1$ then by taking $a_2 = a_1+b_1c_1d_1$ the elements $x_1+x_2+x_3$, a_2, b_1, c_1d_1, $b_1d_1c_1$ form a sublattice isomorphic to M_4 as shown in figure 4.5.

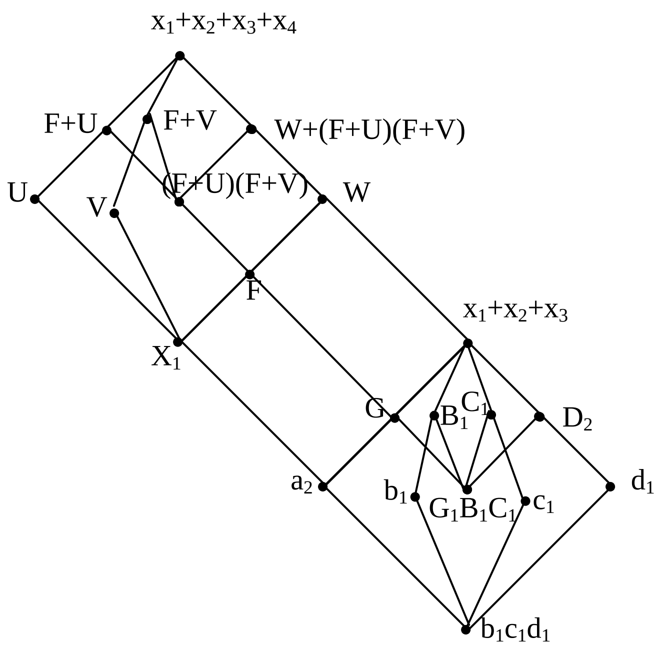


Figure 4.5

In this case, consider $F = X_1 + b_1c_1d_1$ and

$$F\,(x_1+x_2+x_3) = a_2 + x_4\,(x_1 + x_2 + x_3) = G.$$

Let

$$B_1 = b_1 + Gd_1$$
$$C_1 = c_1 + Gd_1$$
$$D_1 = d_1.$$

These satisfy

$$G + B_1 = G + C_1 = G + D_1 = B_1 + D_1 = C_1 + D_1$$
$$D_1G = D_1B_1 = D_1C_1 = d_1G.$$

We have two possibilities

(1) If $GB_1 = GC_1 = B_1C_1 = GB_1C_1$ then $D_2 = D_1 + GB_1C_1$ together with G, B_1, C_1, $x_1 + x_2 + x_3$ and GB_1C_1 gives a sublattice isomorphic to M_4. Also as F lies between X_1 and W.

$(F + U)\ (F + V)$, $(F + U)$, $(F + V)$,

$W + (F + U)\ (F + V)$ and $x_1 + x_2 + x_3 + x_4$

form a sublattice isomorphic to M_3. These two sublattices combine to give us the sublattice isomorphic to $M_{3,4}$ as shown in figure 4.5.

(2) In case one of GB_1, GC_1, B_1C_1 is different from GB_1C_1; let it be $GB_1 \neq GB_1C_1$. Then as shown in lemma of the earlier chapter we obtain a sublattice S_1 isomorphic to $M_{3,3}$ consisting of GB_1C_1, GB_1, $C_1(GB_1+D_1)$, $(GB_1C_1+D_1)(GB_1+C_1)$, $D_2 = (GB_1+C_1)\ (GB_1+D_1)$ $G(B_1+D_2)$, $(G+D_2)B_1$, $(G+D_2)\ (B_1+D_2)$.

Consider the interval $I = (G, G+D_2)$; this interval is projective to $(G(B_1+D_2), (G+D_2)\ (B_1+D_2))$ and so is nontrivial, as otherwise $GB_1 = GB_1C_1$. Now the interval $I+F = (F, F+D_2)$ is a nontrivial subinterval of $J = (F, W)$ as J is projective to $(G, x_1+x_2+x_3)$. Let $I+F = (p, q)$ (say) then the elements $(U+p)\ (V+p)$, $q+(U+p)\ (V+p)$, $(V+q)$, $(U+q)\ (V+p)$, $(U+q)$, $(V+q)$ form a sublattice isomorphic to M_3. This together with the sublattice S_1 (mentioned above) gives us the required sublattice with a homomorphic image isomorphic to $M_{3,3,3}$ as shown in figure 4.6.

If on the other hand one of b_1c_1, c_1d_1, d_1b_1 is other than $b_1c_1d_1$ then without loss in generality, let us assume (2) $b_1c_1d_1 \neq b_1c_1$.

Let $a_1 = a\ (x_1+x_2+x_3)$ then
$a_1+b_1 = a_1+c_1 = a_1+d_1 = b_1+c_1 = b_1+d_1 = c_1+d_1$ and $a_1b_1 = a_1c_1 = a_1d_1$

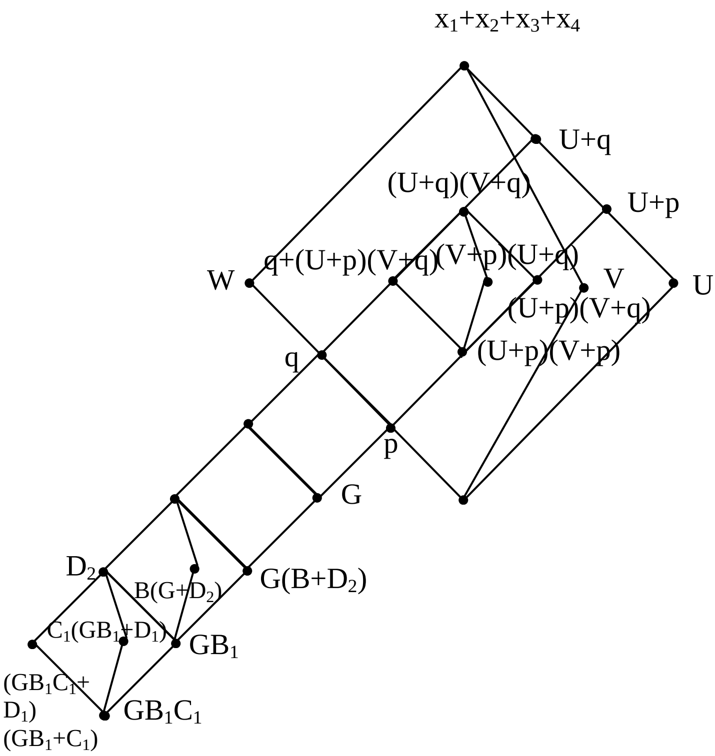


Figure 4.6

Consider the elements $a_1 + b_1d_1 + c_1d_1$, $b_1 + c_1d_1$, $c_1+d_1b_1$, d_1 call them a_2, b_2, c_2, d_2 respectively (say).

Those four elements satisfy the following conditions.

(1) the sum of any 2 of the four equals a single element p and

(2) the product of d_1 with any one of the other three equals a single element q.

Let $G = a_2+x_4(x_1+x_2+x_3)$

$B_1 = b_2 + Gd_2$

$C_1 = c_2 + Gd_2$

$D_1 = d_2$,

then we have a situation exactly similar to what we had in the earlier case. A similar reasoning gives us to required result. Thus we have exhausted all the possibilities under case I.

Next we take up the possibilities under case II. Let us recall that under this case every element of the set T equals $x_1 + x_2 + x_3 + x_4$.

Let T_1 be the set of elements

$$T_1 \quad \begin{cases} x_1 + x_2 + ax_3 + ax_4; ax_1 + ax_2 + x_3 + x_4 \\ x_1 + ax_2 + x_3 + ax_4; ax_1 + x_2 + ax_3 + x_4 \\ ax_1 + x_2 + x_3 + ax_4; x_1 + ax_2 + ax_3 + x_4 \end{cases}.$$

Again two cases may arise.

(A) Each element of T_1 equals $x_1+x_2+x_3+x_4$.

(B) At least one element of T_1 is different from $x_1+x_2+x_3+x_4$.

If (A) is satisfied then set

$$\begin{aligned} \lambda_1 &= x_1 + ax_2 + ax_3 + ax_4 \\ \lambda_2 &= ax_1 + x_2 + ax_3 + ax_4 \\ \lambda_3 &= ax_1 + ax_2 + x_3 + ax_4 \\ \lambda_4 &= ax_1 + ax_2 + ax_3 + x_4 \end{aligned}$$

and $a_1 = a(x_1+x_2+x_3+x_4)$.

The sum of any two of the above equal $x_1+x_2+x_3+x_4$. The product of a_1 with any one of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ equal

$ax_1 + ax_2 + ax_3 + ax_4$.

If (1) $\lambda_i \lambda_j = \lambda_1 \lambda_2 \lambda_3 \lambda_4$ for all $i \neq j$, $i, j = 1, 2, 3, 4$ then the elements a_2 $(=a_1 + \lambda_1 \lambda_2 \lambda_3 \lambda_4)$, $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, $(x_1 + x_2 + x_3 + x_4)$ and $\lambda_1 \lambda_2 \lambda_3 \lambda_4$ form a sublattice isomorphic to M_5 as shown in figure 4.7.

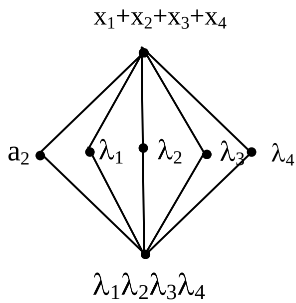


Figure 4.7

If (2) $\lambda_i \lambda_j = \lambda_j \lambda_k = \lambda_k \lambda_i = \lambda_i \lambda_j \lambda_k$ for some $i \neq j \neq k$, $(i, j, k) \in (1, 2, 3, 4)$ and $\lambda_i \lambda_j \lambda_k \neq \lambda_1 \lambda_2 \lambda_3 \lambda_4$ then by a suitable permutation we can assume

$\lambda_1 \lambda_2 = \lambda_2 \lambda_3 = \lambda_3 \lambda_4 = \lambda_1 \lambda_2 \lambda_3$ and
$\lambda_1 \lambda_2 \lambda_3 \neq \lambda_1 \lambda_2 \lambda_3 \lambda_4$.

Now $\lambda_1 \lambda_2 \lambda_3 \lambda_4$, $(a_1 + \lambda_1 \lambda_2 \lambda_3 \lambda_4)$ $(\lambda_1 \lambda_2 \lambda_3 + \lambda_4)$, $\lambda_4 (\lambda_1 \lambda_2 \lambda_3 + a_1) \lambda_1 \lambda_2 \lambda_3$, $(\lambda_1 \lambda_2 \lambda_3 + a_1)$ $(\lambda_1 \lambda_2 \lambda_3 + \lambda_4)$ form a sublattice isomorphic to M_3, while the elements $x_1+x_2+x_3+x_4$, $a_2 = (=a_1 + \lambda_1 \lambda_2 \lambda_3)$, λ_1, λ_2, λ_3, $\lambda_1 \lambda_2 \lambda_3$ form a sublattice isomorphic to M_4.

Now the side $(\lambda_1 \lambda_2 \lambda_3, (\lambda_1 \lambda_2 \lambda_3 + a_1). (\lambda_1 \lambda_2 \lambda_3 + \lambda_4))$ of M_3 above is a subinterval of $(\lambda_1 \lambda_2 \lambda_3, a_2)$ of M_4.

Let $G = (\lambda_1 \lambda_2 \lambda_3 + a_1)(\lambda_1 \lambda_2 \lambda_3 + \lambda_4)$.

Consider G_1 $(G+\lambda_3) \lambda_1$, $(G+\lambda_3) \lambda_2$, λ_3; the product of any two of the above four elements equal $\lambda_1 \lambda_2 \lambda_3$. The sum of λ_3 with any one of the other three equal $G+\lambda_3$.

Now let $H = (G+\lambda_3)\lambda_1$ and $K = (G+\lambda_3) \lambda_2$.

In case $G + H = G + K = H + K = G + H + K$ then $G+H+K$, G, H, K, $\lambda_3 (G+H+K)$, $\lambda_1 \lambda_2 \lambda_3$ form a sublattice isomorphic to M_4; this combines with the M_3 formed by elements $\lambda_1 \lambda_2 \lambda_3 \lambda_4$,

λ_4 (λ_1 λ_2 λ_3 + a_1), (a_1 + λ_1 λ_2 λ_3 λ_4) (λ_1 λ_2 λ_3 + λ_4), λ_1 λ_2 λ_3, G to give a sublattice isomorphic to $M_{4,3}$ as shown in figure 4.8.

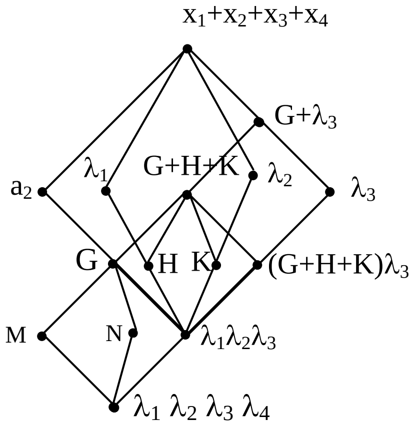


Figure 4.8

In case one of them say G+H $\leq$ G+H+K then G+H+K, (G+H+K) λ_3 + (G+H)K, λ_3(G+H)+K, G+H, λ_3(G+H)+ K(G+H) = D_2, G+HD_2, H+GD_2, HD_2+GD_2 form a sublattice isomorphic to $M_{3,3}$.

Now GD_2 lies between G and λ_1 λ_2 λ_3.

Let M = λ_4 (λ_1 λ_2 λ_3 + a_1)

N = (λ_1 λ_2 λ_3 λ_4 + a_1) (λ_1 λ_2 λ_3 + λ_4).

Then G, GD_2, M+NGD_2, N+MGD_2, MGD_2 + NGD_2 form a sublattice isomorphic to M_3. This sublattice combines with the above mentioned $M_{3,3}$ to give us a sublattice with a homomorphic image isomorphic to $M_{3,3,3}$ as shown in figure 4.9.

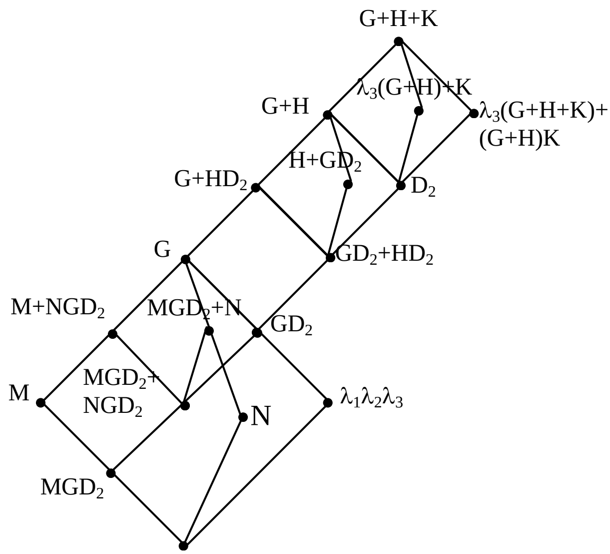


Figure 4.9

Lastly if (iii) $\lambda_1\lambda_2 \neq \lambda_1\ \lambda_2\ \lambda_3 \neq \lambda_1\ \lambda_2\ \lambda_3\ \lambda_4$ then $\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4$, M, N, $\lambda_1\ \lambda_2\ \lambda_3$, G form a sublattice isomorphic to M_3.

Let
$M_1 = \lambda_3\ (\lambda_1\ \lambda_2 + a_1)$
$N_1 = (\lambda_1\ \lambda_2\ \lambda_3 + a_1)\ (\lambda_1\ \lambda_2 + \lambda_3)$
$G_1 = (\lambda_1\ \lambda_2 + a_1)\ (\lambda_1\ \lambda_2 + \lambda_3)$

then $\lambda_1\ \lambda_2\ \lambda_3$, M_1, N_1, $\lambda_1\ \lambda_2$, G form a sublattice isomorphic to M_3.

Further $\lambda_1\ \lambda_2$, λ_1, λ_2, $a_1 + \lambda_1\ \lambda_2$, $x_1+x_2+x_3+x_4$ form a sublattice isomorphic to M_3.

From these three sublattices we extract a sublattice which has a homomorphic image isomorphic to $M_{3,3,3}$. This consists of the elements $\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4$, MN_1, MN_1, $\lambda_1\ \lambda_2\ \lambda_3\ (MN_1 + NN_1)$, $MN_1 + NN_1$; $\lambda_1\ \lambda_2\ \lambda_3$, GN_1, $M_1\ (GN_1 + \lambda_1\ \lambda_2)$, $\lambda_1\ \lambda_2 = (GN_1 + M_1)$, $Q = (GN_1 + \lambda_1\ \lambda_2)\ (GN_1 + M_1)$,

$\lambda_1 \lambda_2$, $(GN_1 + \lambda_1 \lambda_2)$, $M_2 = \lambda_2 (GN_1 + \lambda_1)$, $N_2 = \lambda_1 (GN_1 + \lambda_2)$, $M_2 + N_2$. Figure 4.10

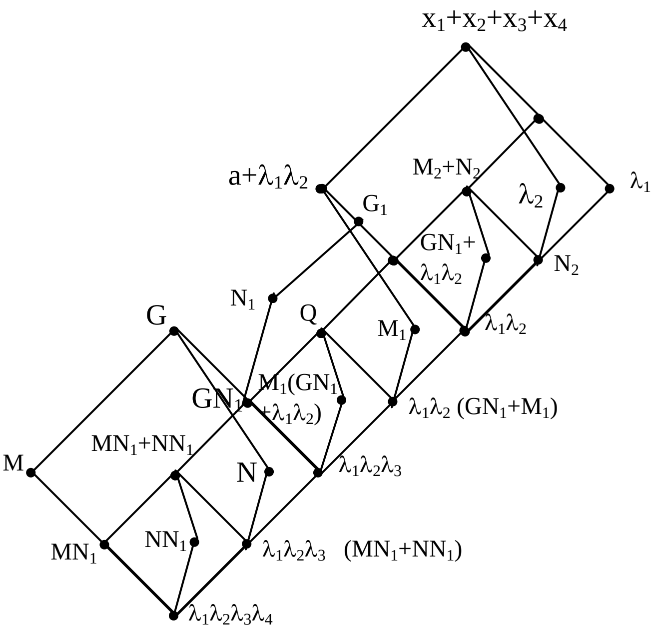


Figure 4.10

So far the case (A). Next we take up case (B).

Recall that in this every element of T equals $x_1+x_2+x_3+x_4$. In particular $x_1 + x_2 + x_3 + ax_4 = x_1 + x_2 + x_3 + x_4$.

This implies

$a(x_1 + x_2 + x_3 + x_4) + x_4 (x_1 + x_2 + x_3) = x_1 + x_2 + x_3 + x_4$ (cf. figure 4.2).

Similarly $x_1 + x_2 + ax_3 + x_4 = x_1 + x_2 + x_3 + x_4$ implies
$a (x_1 + x_2 + x_3 + x_4) + x_3 (x_1 + x_2 + x_4) = x_1 + x_2 + x_3 + x_4$
and the other two equalities yield

$$a(x_1 + x_2 + x_3 + x_4) + x_2(x_1 + x_3 + x_4) = x_1 + x_2 + x_3 + x_4$$

and

$$a(x_1 + x_2 + x_3 + x_4) + x_1 (x_2 + x_3 + x_4) = x_1 + x_2 + x_3 + x_4.$$

Put

$$A = a(x_1 + x_2 + x_3 + x_4)$$
$$X_1 = x_1 (x_2+x_3+x_4)$$
$$X_2 = x_2 (x_1+x_3+x_4)$$
$$X_3 = x_3 (x_1+x_2+x_4)$$
$$X_4 = x_4 (x_1+x_2+x_3)$$

then $A + X_1 = A + X_2 = A + X_3 = x_1 + x_2 + x_3 + x_4$. So $A + X_i > A$ for all $i = 1, 2, 3, 4$.

Now $X_iX_j = x_ix_j$ hence $A > X_i X_j$ $(i \neq j)$. Also $X_i + X_j + X_k$

$$= (x_1 + x_2 + x_3) (x_1 + x_2 + x_4) (x_1 + x_3 + x_4) (x_2 + x_3 + x_4)$$
$$= X_1 + X_2 + X_3 + X_4$$

for all (i, j, k) $(i \neq j \neq k) \in (1, 2, 3, 4)$

If for this set of elements A, X_1, X_2, X_3, X_4 the corresponding set T_1 of elements are all equal, then we proceed as in the case (A). If not then without loss in generality let the element of T_1 other than $X_1 + X_2 + X_3 + X_4$ be $X_1 + X_2 + AX_3 + AX_4$.

Either (1) $A (X_1 + X_2) + X_3 + X_4 \neq X_1 + X_2 + X_3 + X_4$ or (2) $A (X_1 + X_2) + X_3 + X_4 = X_1 + X_2 + X_3 + X_4$.

In case (1) occurs

$$X_1 + X_2 + A(X_3+X_4) \neq X_1 + X_2 + X_3 + X_4$$

for otherwise the upper diamond of figure 4.11 collapses forcing $X_1 + X_2 + X_3 + X_4 = A (X_1 + X_2) + X_3 + X_4$ a contradiction. Thus in this case of elements $X_1 + X_2 + X_3$, $X_1 + X_2 + A (X_3 + X_4)$, $A (X_1+X_2) + X_3 + X_4$,

$A (X_1 + X_2 + X_3) + (X_1 + X_2) (X_3 + X_4)$, $A (X_1 + X_2) + A (X_3 + X_4) + (X_1 + X_2) (X_3 + X_4)$; $X_1 + X_2$, [$A (X_1 + X_2) +$

$(X_1 + X_2)(X_3 + X_4)] = P$, $P(X_1 + AX_2) + (X_2 + AX_1)$, $(X_2 + AX_1)P + (X_1 + A X_2)$,

$R = [P(X_1+AX_2) + P(X_2 + AX_1)]$; $X_3 + X_4$.
$[A(X_3 + X_4) + (X_1 + X_2)(X_3 + X_4)] = Q$,
$(X_3 + AX_4) + Q(X_4 + AX_3)$, $(X_4 + AX_3) + Q(X_3 + AX_4)$,
$S = [Q(X_4 + AX_3) + Q(X_3 + AX_4)]$;

PQ, PS, QR and RS form a sublattice with a homomorphic image isomorphic to $\overset{\frown}{M}_{3,3,3}$ as shown in figure 4.11.

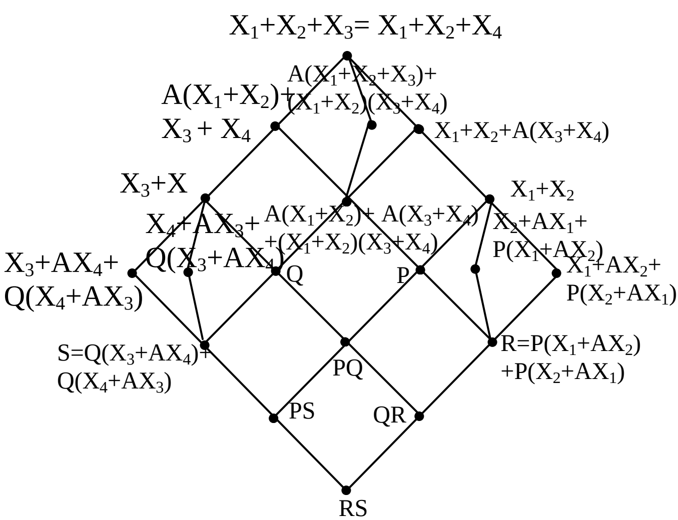


Figure 4.11

In the other case that is in (2) we have
$A(X_1 + X_2) + X_3 + X_4 = X_1 + X_2 + X_3 + X_4$.

Set

$A_1 = A(X_1 + X_2 + X_3 + X_4)$
$B_1 = A(X_1 + X_2) + AX_3 + X_4$
$C_1 = A(X_1 + X_2) + X_3 + AX_4$
$D_1 = X_1 + X_2 + AX_3 + AX_4$.

then the sum of any two of the above four elements equals $X_1 + X_2 + X_3 + X_4$; and the product of A_1 with any one of the other three equals $A(X_1 + X_2) + AX_3 + AX_4$.

If $B_1 C_1 = C_1 D_1 = D_1B_1$ we obtain $M_{4,3}$ as shown in figure 4.12.

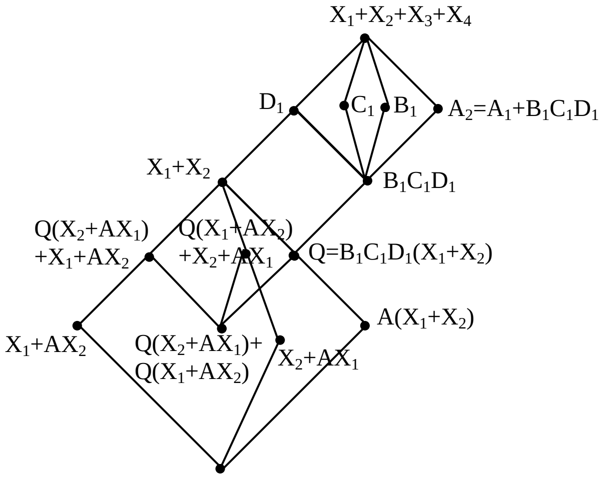


Figure 4.12

This consists of the elements $X_1 + X_2 + X_3 + X_4$, $A_2 = A_1 + B_1C_1D_1$, B_1, C_1, D_1, X_1+X_2, $Q = B_1 C_1 D_1 (X_1 + X_2)$, $(X_1 + AX_2) + Q(X_2 + AX_1)$, $Q(X_1 + AX_2) + (X_2 + AX_1)$, $Q (X_1 + AX_2) + Q(X_2 + AX_1)$.

If $B_1C_1 \neq B_1C_1D_1$ then we obtain $M_{3,3,3}$ as shown in figure 4.13. This consists of the elements $B_1C_1D_1$, $D_1(A_1+B_1C_1)$, $(A_1+B_1C_1D_1) (D_1+B_1C_1)$, B_1C_1, $D_2 = (D_1 + B_1C_1)(A_1 + B_1C_1D_1)$, $C_1 (B_1+D_2)$, $B_1 (C_1+D_2)$, $(B_1+D_2) (C_1+D_2)$.

$R = Q (AX_1 + X_2) + Q (X_1 + AX_2)$ (where
$Q = D_1 (A_1 + B_1C_1) (X_1 + X_2)$),

$RA(X_1 + X_2)$, $Q(AX_2 + X_1)$, $Q(X_2 + AX_1)$ and $AX_1 + AX_2$.

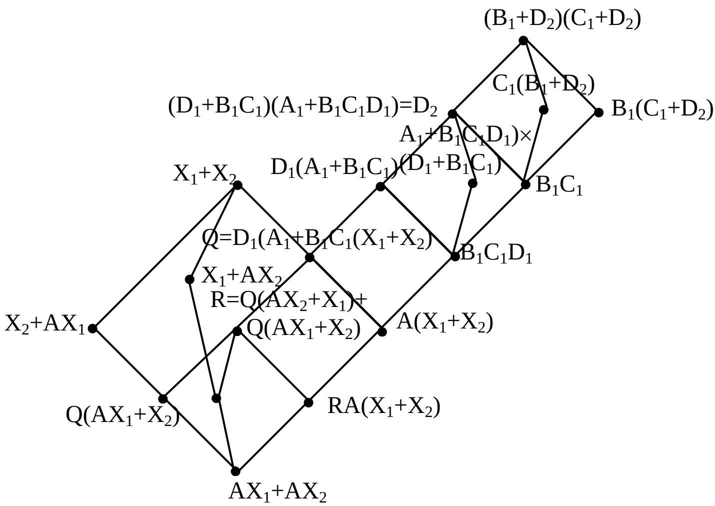


Figure 4.13

Thus we conclude this analysis by giving the theorem characterizing semi-supermodular lattices.

THEOREM 4.2: *A modular lattice L is semi-supermodular if and only if it does not contain any sublattice, whose homomorphic image is isomorphic to* $M_{3,3,3,}$ $M_{4,3}$, $M_{3,4}$, M_5 *or* $\widehat{M}_{3,3,3}$.

Dually we observe

THEOREM 4.3: *The dual of a modular lattice L is semi-super modular if and only if it does not contain any sublattice whose homomorphic image is isomorphic to* $M_{3,3,3}$, $M_{4,3}$, $M_{3,4,}$ M_5 *and* $\breve{M}_{3,3,3}$ *(*$\breve{M}_{3,3,3}$ *is the dual of the lattice* $\widehat{M}_{3,3,3}$*). (cf. figure 4.1).*

Hence we have.

THEOREM 4.4: *The dual of a semi-supermodular lattice L is not necessarily semi-supermodular.*

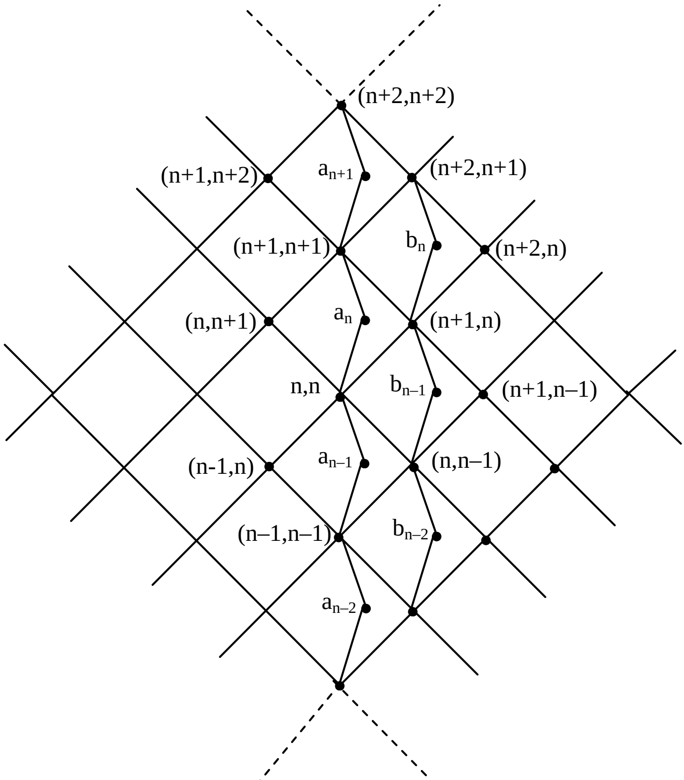


Figure 4.14

***Proof*:** By an example. Observe that $\breve{M}_{3,3,3}$ is semi supermodular; while its dual $\hat{M}_{3,3,3}$ is not.

Unlike the equational class of supermodular lattices, the equational class of semisuper modular contains infinite subdirectly irreducible members.

THEOREM 4.5: *The equational class of semi-supermodular lattices is not generated by its finite members.*

***Proof*:** It will suffice to give an example of an infinite subdirectly irreducible semi-supermodular lattice.

Consider the lattice L in figure 4.14. L consists of the direct product of Z (the set of all integers +ve, –ve, 0) with itself with the usual natural direct product order.

For each integer n, two, extra element a_{n-1}, b_{n-1} are inserted in the squares formed by [((n–1), (n–1)), (n, n–1), (n–1, n), (n, n)] and [(n, n–1), (n+1, n–1), (n, n) (n+1, n)] so as to make them sublattices isomorphic to M_3 (see figure 4.14).

Observe that every prime interval of L is projective to any other prime interval of L. Thus L is subdirectly irreducible as it is discrete. Further L has no sublattice with a homomorphic image isomorphic to $M_{3,3,3}$, $M_{3,4}$, $M_{4,3}$ or M_5 or $\overset{\frown}{M}_{3,3,3}$. Hence L is semi-supermodular. L is obviously infinite. Hence the statement of the theorem.

Now before we generalize semi-supermodularity for any integer n, we observe the following facts, which we enunciate in the form of the following lemma.

Lemma 4.4: Let $x_1, x_2, \ldots, x_n$ be a set of n elements (n > 3) of a modular lattice L satisfying

$$x_i + x_j = 1 \text{ for } i \neq j \text{ for all } i, j = 1, 2, \ldots, n.$$
$$x_i x_j = 0 \text{ for } i \neq j \text{ for all } i, j = 1, 2, \ldots, n.$$

that is they form a sublattice isomorphic to M_n.

(1) Let a_1 be an element between x_1 and 1, then there exist elements a_i between x_i and 1 for all i = 2, 3, …, n such that the sublattice of L generated by the set of n elements $a_1, a_2, \ldots, a_n$ contains a homomorphic image isomorphic to

$$M_{i_1+2, i_2+2, \ldots, i_r+2} \quad (\text{where } i_1 + i_2 + \ldots + i_r = (n-1)$$

$$\text{and } i_1, i_2, \ldots, i_r \geq 1).$$

(2) Let b_1 be an element between 0 and x_1, then there exist element b_1 between 0 and x_i for each i = 2, 3, …, n, such that the sublattice of L generated by the set of n elements $b_1, b_2, \ldots, b_n$ contains a homomorphic image isomorphic to $M_{i_1+2, i_2+2, \ldots, i_r+2}$ (where $i_1 + i_2 + \ldots + i_r = (n-2)$ and $i_1, i_2, \ldots, i_r \geq 1$).

Further a subinterval of $(1, a_1)$ or $(0, b_1)$ is projective with the essential side of $M_{i_1+2, i_2+2, \ldots, i_s+2}$ so obtained.

Proposition 4.1: *Essential side of* $M_{i_1+2, i_2+2, \ldots, i_r+2}$ *is the side, whose additive or multiplicative translate meets every one of the* M_i*'s involved in the figure of* $M_{i_1+2, i_2+2, \ldots, i_r+2}$.

***Proof*:** By mathematical induction on the integer n. When n = 3. The elements a_1, $a_2 = x_2 + a_1x_3$; $a_3 = x_3 + a_1x_2$, $a_1x_2 + a_1x_3$ and 1 form a sublattice isomorphic to M_3 with $(a_1, 1)$ as a side with the elements b_1, $b_2 = x_2\ (a_1 + x_3)$, $b_3 = x_3\ (a_1+x_2)$, $(a_1+x_2)\ (a_1+x_3)$, 0 form a sublattice isomorphic to M_3 with $(0, b_1)$ as a side. Thus when n = 3 the result is true.

Assume the result to be true for all $n \leq m-1$. To prove it for n = m. Set

$$
\begin{aligned}
a_2 &= a_1x_n + x_2 \\
a_3 &= a_1x_n + x_3 \\
&. \\
&. \\
&. \\
a_n &= x_n
\end{aligned}
$$

then $a_i + a_j = 1$ for all i, j = 1, 2, …, n, $a_ia_n = a_ix_n$ for all i = 1, 2, …, n–1.

(1) if $a_i\ a_j = a_1a_2 \ldots a_{n-1}$ for all i, j = 1, 2, …, n–1 $(i \neq j)$ then 1, $a_1, a_2, \ldots, a_{n-1}, a_n+a_1a_2 \ldots a_{n-1}, a_1a_2, \ldots, a_{n-1}$ form a sublattice isomorphic to M_n.

(2) If $a_1 a_2 \dots a_{n-1} \neq a_{\sigma_1} \dots . a_{\sigma_{n-2}}$ for a permutation ($\sigma_1, \sigma_2, \dots, \sigma_{n-2}, \sigma_{n-1}$) of (1, 2, 3, …, n–1) then we can rearrange 1, 2, …, n–1 and get it as $a_1a_2 \dots a_{n-1} \neq a_1a_2 \dots a_{n-2}$.

If further $a_ia_j = a_1a_2 \dots a_{n-2}$ for all i, j = 1, 2, …, (n–2) ($i \neq j$) then we obtain the two sublattices M_3 formed by

$a_1a_2 \dots a_{n-1}$, $a_1a_2 \dots a_{n-2}$, $c_n = a_{n-1} (a_1a_2 \dots a_{n-2} + a_n)$, $d_n = (a_1a_2 \dots a_{n-1} + a_n) (a_1 \dots a_{n-2} + a_{n-1})$, $(a_1a_2 \dots a_{n-2} + a_n) (a_1a_2 \dots a_{n-2} + a_{n-1}) = b_n$ and M_{n-1} formed by 1, a_1, a_2, …, a_{n-2}, $a_n + a_1a_2 \dots a_{n-2}$, $a_n + a_1a_2 \dots a_{n-2}$, $a_1a_2 \dots a_{n-2}$.

Now b_n lies between $(a_1a_2 \dots a_{n-2} + a_n)$ and 1. By induction hypothesis applied to b_n for this M_{n-1} we get a sublattice isomorphic to $M_{i_1+2, i_2+2, \dots, i_k+2}$ with $i_k > 1$ and $i_1 + i_2 + \dots + i_k = (n-3)$.

Further, the essential side of this $M_{i_1+2, i_2+2, \dots, i_k+2}$ is projective with a subinterval of $(a_1a_2 \dots a_{n-2}, b_n)$; which helps us to be an adjoining M_3. Hence we obtain $M_{i_1+2, i_2+2, \dots, i_k+2, 3}$ with $i_k \geq 1$ and $i_1 + \dots + i_k + 1 = (n-2)$ giving us the required result.

(3) if $a_1a_2 \dots a_{n-2} \neq a_1a_2 \dots a_{n-3}$ but $a_1a_2 \dots a_{n-3} = a_ia_j$ for all i, j = 1, 2, …, n–3. Set $e_n = (a_1a_2 \dots a_{n-2} + a_n) (a_1 \dots a_{n-3} + a_{n-2}) (a_1a_2 \dots a_{n-2} + a_{n-1})$ then we obtain a M_3 consisting of $c_ne_n + d_ne_n$, c_ne_n, $d_nc_n\ a_1 \dots a_{n-2} (c_n e_n + d_ne_n)$, $a_1a_2 \dots a_{n-1}$, another M_3 consisting of $a_1a_2 \dots a_{n-2}$, $e_n (e_n+c_{n-1})\ a_1a_2 \dots a_{n-3}$, $(e_n + a_1a_2 \dots a_{n-3})\ c_{n-1}$, $(e_n + c_{n-1}) (e_n+a_1a_2 \dots a_{n-3})$ such that the side

$c_ne_n + d_ne_n$, $a_1 \dots a_{n-2} (c_ne_n + d_ne_n)$ of the first M_3 is projective with the side $(e_n, a_1 \dots a_{n-2})$.

Also the side $[(e_n+c_{n-1}) (e_n + a_1a_2 \dots a_{n-3}), (e_n+c_{n-1})\ a_1 \dots a_{n-3}]$ of the lattice M_3 is projective with the interval $(a_1a_2 \dots a_{n-3}, e_n + a_1 \dots a_{n-3})$.

Put $e_n + a_1 \ldots a_{n-3} = f$; f is an element between $a_1 \ldots a_{n-3}$ and $a_1a_2 \ldots a_{n-3} + a_n$ and the elements 1, $a_1, \ldots, a_{n-3}$, $a_n + a_1 \ldots a_{n-3}$, $a_1a_2 \ldots a_{n-3}$ form a sublattice isomorphic to M_{n-2}.

Again the induction hypothesis gives us a, $M_{i_1+2, i_2+2, \ldots, i_l+2}$ such that $i_1 + \ldots + i_l = (n-4)$. These M_i's together with the $M_{3,3}$ mentioned above gives us the required M.

Thus we proceed until we exhaust all the a_i's and come to a_1a_2 alone.

Now we proceed to generalize the notion of semi-supermodularity.

DEFINITION 4.2: *A lattice L is called n-semi supermodular if it satisfies the identity*

$(a+a_1) \ldots (a+a_n)$

$$= a + \sum_{\substack{i \neq j \\ i,j=1}}^{n} a_i a_j (a+a_1)\ldots(a \hat{+} a_i)(a \hat{+} a_j)\ldots(a+a_n)$$

(where $(a \hat{+} a_i)$ *means* $(a+a_i)$ *is omitted).*

Just as we have proved the results for semi-supermodular lattices we can obtain the following results for any n-semi supermodular lattices.

Lemma 4.5: Any n-semi-supermodular lattice is modular.

THEOREM 4.6: *A modular lattice L is n-semi-supermodular if and only if there does not exist a set of (n+1) elements* $a, a_1, \ldots, a_n$ *in L such that* $a + a_1 = a + a_2 = \ldots + a + a_n > a$ *with* $a > a_ia_j$ $(i \neq j)$ $(i, j = 1, 2, \ldots, n)$.

THEOREM 4.7: *A modular lattice L is n-semi-supermodular if and only if it does not contain a sublattice whose homomorphic images is isomorphic to* $M_{i_1+2, i_2+2, \ldots, i_r+2}$ *or* $\widehat{M}_{i_1+2, i_2+2, \ldots, i_r+2}$ *with* $i_1 + i_2 + \ldots + i_r = n-1,\ i, j \geq 1.$

***Proof*:** We essentially use lemma and adopt a similar method of proof as in the case of Theorem 4.2.

Corollary 4.1: The dual of a n-semi supermodular lattice is not necessarily n-semi-super modular.

Corollary 4.2: Any m-semi-supermodular lattice is n-semi-supermodular when m ≤ n. Corollary 4.2 combined with Theorem 4.5 gives;

Corollary 4.3: The equational classes of n-semisupermodular lattices (for n ≥ 4) are not generated by their finite members.

Chapter Five

SOME INTERESTING EQUATIONAL CLASSES OF MODULAR LATTICES

In chapter III, we studied supermodular lattices and characterized them as those modular lattices which do not contain sublattices isomorphic to M_4 or $M_{3,3}$. In this chapter we are interested in characterizing those lattices which do not contain sublattices isomorphic to M_4 alone (cf. theorem 5.1 and 5.2). Lastly we define modular elements in a general lattice L and characterize them in theorem 5.3.

We start with the investigation of the following lattice identity (A).

$(a+b)\ (a+c)\ (a+d)\ (ab+(a+b)\ c+d))\ (ac+(a+c)\ d+b) \times\ (ad + (a+d)\ b+c) = ab + ac + ad + (a+b)\ (ab+c)d + (a+c)\ (ac+d)b + (a+d)\ (ad+b)c$ for all a, b, c, d in L.

Remark: In any lattice the left hand side of equality (A) is in general greater than or equal to the right hand side of (A).

Lemma 5.1: Any lattice L satisfying identity (A) is modular.

***Proof*:** Let L be a lattice satisfying identity (A); put b = c in (A) we get (a+d) (b+ad) = ad + b(a+d) for all a, b, d in L which can easily be recognized as the modular law. Hence L is modular. The converse however is not true.

Thus we prove in the following:

Lemma 5.2: Every modular lattice need not necessarily satisfy identity (A).

***Proof*:** By an example.

Consider the lattice L of figure 3.1. It does not satisfy the identity (A) for the set (a, b, c, d) of elements as specified in M_4. That is, in this case the left hand side of (A) equals 1 and the right hand side equal 0.

As an immediate corollary, we obtain.

Corollary 5.1: If L is any lattice satisfying identity (A) then it cannot contain a sublattice isomorphic to M_4.

Proof is an easy consequence of the fact that any sublattice S of a lattice L satisfying identity (A) also satisfies (A).

Now arises the natural questions whether every modular lattice not satisfying identity (A) contains a sublattice isomorphic to M_4?

Fortunately for us, we again have a counter example.

Lemma 5.3: A modular lattice L not satisfying identity (A) need not necessarily contain a sublattice isomorphic to M_4.

***Proof*:** Consider the lattice of figure 5.1.

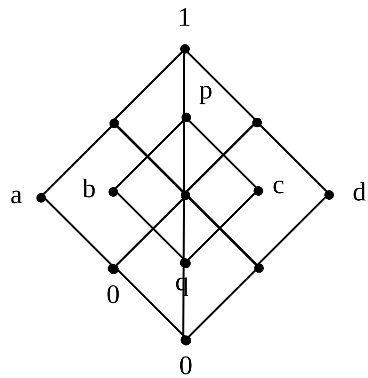


Figure 5.1

and the elements a, b, c, d as marked in the figure 5.1. The left hand side of identity (A) equals p, while the right hand side equals q. Hence L does not satisfy identity (A). One can easily see from the figure the absence of any sublattice in L, isomorphic to M_4.

Next we characterize all modular lattices which satisfy identity (A) in the following theorem.

THEOREM 5.1: *A modular lattice L does not satisfy identity (A) if and only if it contains a non-distributive triple (b, c, d) such that b, c, d are the common relative complements of an element 'a' in some interval of L.*

[(b, c, d) is called a non-distributive triple if the lattice generated by the triple is not distributive].

***Proof*:** Let L be a modular lattice containing a non-distributive triple (b, c, d) such that b, c, d are the common relative complements of a single element a in L in the interval (y, x).

That is

$$a+b = a+c = a+d = x$$
$$ab = a.c = a.d = y \text{ and}$$
$$(b+c)\ (b+d)\ (c+d) > bc+cd+db.$$

Now the set of four elements (a, b, c, d) does not satisfy identity (A) in L. For when these elements are substituted in identity (A) the left hand side equals

$$x.x.x\ (y+xc+d)\ (y+xd+b)\ (y+xb+c)$$
$$= x(c+d)\ (d+b)\ (b+c) \quad (\text{as } x > b, c, d > y)$$
$$= (c+d)\ (d+b)\ (b+c) \quad (\text{as } x > b+c).$$

The right hand side equals

$$y + y + y + x\,(y + c)d + x(y + d)b + x\,(y + b)c$$
$$= y + cd + db + bc \qquad (\text{as } x > b, c, d > y)$$
$$= cd + db + bc \qquad (\text{as } bc > y).$$

By hypothesis (b, c, d) is a non-distributive triple. Hence the two sides are not equal. Conversely let L be a modular lattice not satisfying identity (A). That is there exists a set of four elements (a, b, c, d) in L such that

$$(a+b)\ (a+c)\ (a+d)\ (ab+(a+b)c+d)\ (ac+(a+c)d+b)(ad+(a+d)b+c) \neq$$
$$ab+ac+ad+(a+b)\ (ab+c)d +$$
$$(a+c)\ (ac+d)b + (a+d)\ (ad+b)c. \qquad (1)$$

$$\text{Let } e_1 = (a+b)\ (a+c)\ (ab+ac+d)$$
$$= ab + ac + d\ (a+b)\ (a+c)$$
$$(L \text{ is modular})$$

$$e_2 = (a+c)\ (a+d)\ (ac+ad+b)$$
$$= ac + ad + b(a+c)\ (a+d)$$

$$e_3 = (a+d)\ (a+b)\ (ad+ab+c)$$
$$= ad+ab+c(a+d)\ (a+b).$$

Then $a + e_1 = a + e_2 = a + e_3 = (a+b)(a+c)(a+d)$ and $ae_1 = ae_2 = ae_3 = ab + ac + ad$.

$$
\begin{aligned}
e_1 + e_2 &= ab + ac + d(a+b)(a+c) + ac + ad + b(a+c)(a+d) \\
&= ab + ac + ad + (a+b)(d(a+c) + b(a+c)(a+d)) \\
&\qquad\qquad (\text{as } (a+b) \geq b(a+c)(a+d)) \\
&= ab+ac+ad + (a+b)(a+c)(d(a+c) + b(a+d)) \\
&\qquad\qquad (\text{as } (a+c) \geq d(a+c)) \\
&= ab+ac+ad+(a+b)(a+c)(a+d) \quad (d(a+c)+d) \\
&\qquad\qquad (\text{as } (a+d) \geq d(a+c)) \\
&= (a+b)(a+c)(a+d)(ab+ac+ad+b+d(a+c)) \\
&\qquad\qquad (\text{as } (a+b)(a+c)(a+d) \geq ab+ac+ad) \\
&= (a+b)(a+c)(a+d)(ac+d(a+c) + b) \\
&\qquad\qquad (\text{as } ab < b \text{ and } ad < (a+c)d).
\end{aligned}
$$

Similarly we obtain

$$
\begin{aligned}
e_2 + e_3 &= (a+b)(a+c)(a+d)(ad+(a+d)b+c) \quad \text{and} \\
e_3 + e_1 &= (a+b)(a+c)(a+d)(ab + (a+b)c+d)
\end{aligned}
$$

By symmetry of the operations (+) and (.) in this set up, we have

$$
\begin{aligned}
e_1e_2 &= ab + ac + ad + (a+c)(ac+d)b \\
e_2e_3 &= ab + ac + ad + (a+d)(ad+b)c \\
e_3e_1 &= ab + ac + ad + (a+b)(ab+c)d.
\end{aligned}
$$

Hence, the left hand side of ineuqatity (1) equals $(e_1+e_2)(e_2+e_3)(e_3+e_1)$ and the right hand side equals $e_1e_2 + e_2e_3 + e_3e_1$.

Hence, we establish the existence of a non-distributive triple (e_1, e_2, e_3) in L, this is turn implies no $e_i = e_j$ for $i \neq j$ (i, j = 1, 2, 3) as otherwise (e_1, e_2, e_3) will cease to be non-distributive.

Also $a \neq e_i$ for any $i = 1, 2, 3$; otherwise if $a = e_i$ then $a + e_i = ae_i$. This implies $(a+b)(a+c)(a+d) = ab+ac+ad$.

Now $(a+b)(a+c)(a+d) \geq e_1, e_2, e_3 \geq ab + ac + ad$. So the sublattice generated by (e_1, e_2, e_3) is contained in the convex sublattice.

$$S = (ab + ac + ad, (a+b)(a+c)(a+d))$$

In particular $(e_1 + e_2)(e_2 + e_3)(e_3 + e_1)$ and $e_1e_2 + e_2e_3 + e_3e_1$ are elements in S. If the greatest and the least elements of S are equal then any two of the elements in the convex sublattice S are equal. Thus the two elements

$$(e_1 + e_2)(e_2 + e_3)(e_3 + e_1) \text{ and } e_1e_2 + e_2e_3 + e_3e_1$$

are equal.

A contradiction to our hypothesis. Hence we establish the existence of a non-distributive triple (e_1, e_2, e_3) in L such that they are the common relative complements of the single element a in the interval $(ab + ac + ad, (a+b)(a+c)(a+d))$.

Lastly we come to the characterization of those modular lattices which contain a sublattice to M_4 in the following.

THEOREM 5.2: *A modular lattice L contains a sublattice isomorphic to M_4 if and only if L contains a subset of four elements (a, b, c, d) such that equality (A) is not satisfied by this set. However this set of four elements satisfy the following identity and its dual for every permutation of the element (b, c, d).*

$$\begin{gathered} a\,(ab+(a+b)c+d)\,(ac+(a+c)\,d+b))\,(ad+(a+d)\,b+c) + \\ (a+b)\,(a+c)\,(ab+ac+d)\,(ad+(a+d)\,b+c) \end{gathered}$$

$$\begin{gathered} = (a+b)\,(a+c)\,(a+d)\,(ab+(a+b)\,c+d) \times (ac + (a+c)\,d+b) \\ (ad + (a+d)\,b+c) \qquad \ldots\ (B) \end{gathered}$$

***Proof*:** Let L be a lattice containing a sublattice S isomorphic to M_4 consisting of 1, a, b, c, d, 0. Then the set of four elements of L corresponding to the four elements (a, b, c, d) of M_4 satisfy the requirements of the theorem.

Conversely if L contains a set of four elements (a, b, c, d) not satisfying (A) then from the Theorem 5.1 we see that the triple (e_1, e_2, e_3) is a non-distributive triple, where e_1, e_2, e_3 are as defined in the proof of Theorem 5.1. Further we observe that the equalities mentioned in Theorem 5.2 are precisely the following equalities.

$ap + e_ip = p(a+q)\,(e_i+q) = q$ for i = 1, 2, 3 where
$p = (e_1 + e_2)\,(e_2+e_3)\,(e_1+e_3)$ and $q = e_1e_2 + e_2e_3 + e_3e_1$.

Now $(q+ap,\ q+e_1p,\ q+e_2p,\ q+e_3p)$ generate a sublattice isomorphic to the lattice of figure 5.1. For

$q + e_1p = e_1\,(e_2 + e_3) + e_2e_3$
$q + e_2p = e_2\,(e_1 + e_3) + e_1e_3$
$q+e_3p = e_3\,(e_1+e_2) + e_1e_2.$

So the sum of any two of these is p and the product of any two of these is q. Hence they are not equal as p is not equal to q.

Also

$$\begin{aligned} q + ap + q + e_ip &= q + ap + e_ip \\ &= q+p = p \end{aligned}$$

and

$$\begin{aligned} (q+ap)\,(q+e_ip) &= p\,(q+a)\,(q+e_i) \\ &= pq = q. \end{aligned}$$

So $ap + q$ does not equal $q + e_ip$ for i = 1, 2, 3.

DEFINITION 5.1: *An element a of a lattice L is called modular if and only if the sublattice by (a, b, c) for all b, c in L is modular.*

THEOREM 5.3: *An element a of a lattice L is modular if and only if*

(1) $(a+b)(a+c) = a+b(a+c)$ for all b, c in L.
(2) $ab + ac = a(b+ac)$ for all b, c in L.
(3) $a+b = a+c$; $ab = ac$; $b \geq c$ implies $b = c$ for all b, c in L.
(4) $(b+a)(b+c) = b+c(a+b)$ for all b, c in L.
(5) $ba + bc = b(c+ab)$ for all b, c in L.

***Proof*:** Let a be a modular element in a lattice L and let b, c be two arbitrary elements of L. The sublattice S generated by {a, b, c} is modular (by definition). Further all the elements in equalities (1) to (5) belong to S. These equalities are satisfied; as S is modular.

Conversely, let a be an element of a lattice L satisfying equalities (1) to (5) and b, c two arbitrary elements in L. Consider the sublattice generated by S = {a, b, c} in L.

First we observe
if $x \leq y$ in S then $(a+x)y = ay+x$ (6)

For if $p = (a+x)y \leq ay + x = q$ then
$a+p = a+q = a+x$
$ap = a+q = a+x$
$ap = aq = ay$ foreing $p = q$ by (3)

As particular cases of (6) we obtain

$(b+a)(b+c) = b+a(b+c)$ (4′)

$ba + bc = b(a+bc)$ (5′)

Next $(a+b)(a+c)(b+c) = (a+b(a+c))(b+c)$ (by (1))

$= a(b+c) + b(a+c)$ (by (6)).

Similarly

$(a + b)(a + c)(b + c) = a(b + c) + c(a + b)$.

Next $p = (b+ac)(a+c) \geq (a+c)b + ac = q$

$$\begin{aligned} a+p &= a+ (b+ac)(a+c) \\ &= (a+b+ac)(a+c) && \text{(by (1))} \\ &= (a+b)(a+c). \end{aligned}$$

$$\begin{aligned} a+q &= a + (a+c)b \\ &= (a+b)(a+c) && \text{(by (1))} \end{aligned}$$

$$\begin{aligned} ap &= a(b+ac) = ab + ac \\ aq &= a((a+c)b+ac) = a(a+c)b + ac \\ &= (ab+ac) && \text{(by (2))} \end{aligned}$$

Therefore $a+p = a+q$ and $ap = aq$ forcing $p=q$ (by (3)).

That is

$$(b+ac)(a+c) = (a+c)b + ac \qquad (7)$$

Similarly

$$(c+ab)(a+b) = (a+b)c + ab \qquad (8)$$

Also $(a+bc)(b+c) = a(b+c) + bc$ (by (6))

Finally adding (7) and (8) we get

$(b+ac)(a+c) + (c+ab)(a+b)$

$= (a+c)b + (a+b)c$

$$\begin{aligned} \text{L.H.S.} &= (a+c)(b+ac+(c+ab)(a+b)) && \text{(by (4))} \\ &= (a+c)(b+(c+ab)(a+b)) && (ac \leq (c+ab)(a+b)) \\ &= (a+c)(b+c(a+b) + ab) && \text{(by (8))} \\ &= (a+c)(b+c(a+b)) \\ &= (a+c)(b+c)(a+b) && \text{(by (4)).} \end{aligned}$$

So $(a+c)(b+c)(a+b) = b(a+c) + c(a+b)$ (9)

The equalities due to (6) - (9) are got dually. Hence the sublattice generated by {a, b, c} in L is a homomorphic image of the free modular lattice generated by (a, b, c); and hence is modular.

As in the case of neutral elements we can show.

THEOREM 5.4: *The set of modular elements of a lattice L is the intersection of the maximal modular sublattice of L.*

***Proof*:** On the same lines as Birkhoff. 1948.

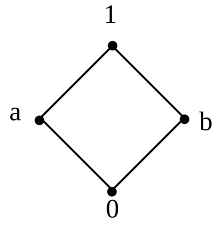
Corollary 5.2: The modular elements of any lattice L form a modular sublattice of L.

Chapter Six

SMARANDACHE LATTICES

In this chapter we for the first time introduce the notion of Smarandache lattices and give a few of its properties. For more about lattices please refer [2].

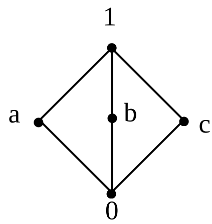
DEFINITION 6.1: *Let L be a lattices; if L has atleast one sublattice whose homomorphic image is isomorphic to the Boolean algebra of order four (i.e., isomorphic to*



then we call L to be a Smarandache lattice (Smar lattice).

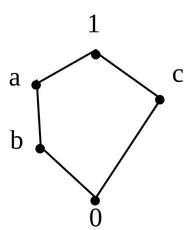
We first illustrate this by some examples.

Example 6.1**:** Let L be a diamond lattice.



L is a Smar lattice. For the sets {0, 1, a, b}, {0, 1, a, c} and {1, 0, b, c} are sublattices whose homomorphic image is isomorphic to the Boolean algebra of order four.

Example 6.2**:** Let L be the pentagon lattice given by



L is a Smar Lattice for the subsets {0, 1, a, c} and {0, 1, b, c} are sublattices of L whose homorphic image is isomorphic to the Boolean algebra of order four.

We see the pentagon lattice has also a sublattice of order four given by the set {1, a, b, 0} whose homomorphic image is not isomorphic with the Boolean algebra of order four.

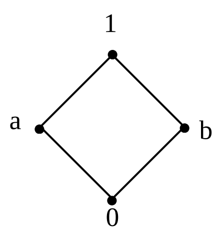
Further both the diamond lattice and pentagon lattice have only three sublattice of order four, but in case of diamond lattice we see all sublattices of order four are isomorphic with the Boolean algebra of order four, however in case of pentagon lattice we see only two of the sublattices are such that their homomorphic image is isomorphic to the Boolean algebra of order four but one sublattice is just a chain lattice of order four.

It is also interesting to note both the pentagon lattice and the diamond lattice are not distributive but both are Smar-lattice.

We see not all lattices are Smar-lattices. Infact we have a class of lattices which are not Smar-lattices.

THEOREM 6.1: *Let L denote the class of all chain lattices; no lattice in L is a Smar lattice.*

***Proof*:** If M in L has to be a Smar-lattice we need a proper subset S in M of order four such that S is a sublattice of M and S is isomorphic with the Boolean algebra of order four.

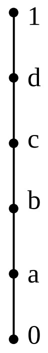


This is turn implies S in L should be such that, S has two distinct elements a, b other than 0 and 1 such a + b = 1 and a.b = 0.

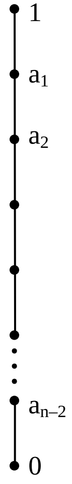
In a chain lattice L we know there does not exists distinct elements a, b ($a \neq b$) a, b $\in L \setminus \{1, 0\}$ such that a+b = 1 and a.b = 0. So S cannot be a sublattice whose homomorphic image is isomorphic with the Boolean algebra of order four.

We give some examples before we proceed onto define or discuss more properties about Smar lattices.

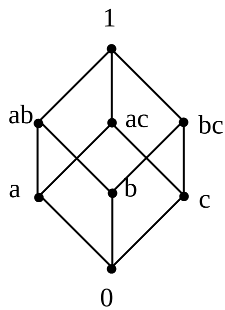
***Example 6.3*:** Let L be a chain lattice of order six. L is not a Smar lattice.



***Example 6.4*:** Let L be a chain lattice of order n. L is not a Smar lattice.

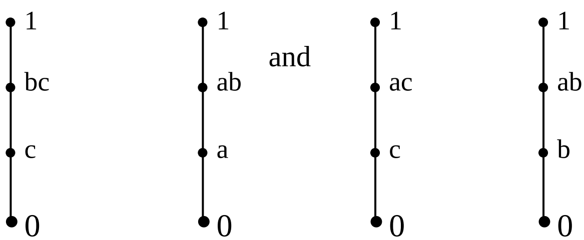


***Example 6.5*:** Let L be a Boolean algebra of order eight given by the following Hasse diagram.

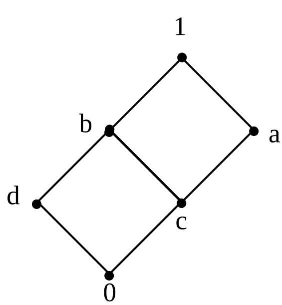


We see this has six sublattices of order four which are such that they are isomorphic with the Boolean algebra of order four. These sublattices are {a, b, 0, ab}, {0, b, c, bc}, {ac, bc, c, 1}, {1, ab, ac, a}, {0, a, c, ac} and {1, ac, bc, b}.

In fact L has also sublattices of order four which are chain lattices.



***Example 6.6*:**



Let L be a lattice given by the Hasse diagram. L is a Smar Lattice for some of the subsets $S_1 = \{b, c, d, 0\}$, $S_2 = \{1, b, a, c\}$, $S_3 = \{1, d, a, 0\}$, $S_4 = \{1, d, c, 0\}$ are sublattices of L whose homomorphic image is isomorphic with a Boolean algebra of order four.

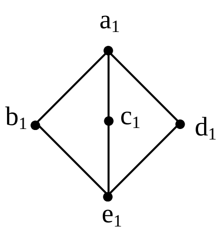
Take $P_1 = \{1, a, c, 0\}$ and $P_2 = \{1, b, d, 0\}$ subsets of L.

These are also sublattices of L of order four but are not isomorphic with chain lattices or order four.

We now proceed onto prove the following interesting result.

THEOREM 6.2: *If L is a modular lattice then L is a Smar Lattice.*

***Proof*:** Given L is a modular lattice, so in L we have triples, a, b, c such that the modular law is satisfied. Thus L has sublattices of order five which are isomorphic with the diamond lattice say we have $S = \{a_1, b_1, c_1, d_1, e_1\} \subseteq L$ such that S has Hasse diagram,



Now clearly $\{a_1, b_1, c_1, e_1\}$ $\{a_1, c_1, d_1, e_1\}$ and $\{a_1, b_1, d_1, e_1\}$ are subsets of S whose homomorphic image is isomorphic to the Boolean algebra of order four. Thus every modular lattice is a Smar lattice.

Now a natural question would be "Is every distributive lattice a Smar lattice?" This question is partially answered by the following Theorem.

THEOREM 6.3: *Every distributive in general is not a Smar lattice.*

***Proof*:** We see all chain lattices are distributive. But none of the chain lattice is a Smar lattice. Hence the claim.

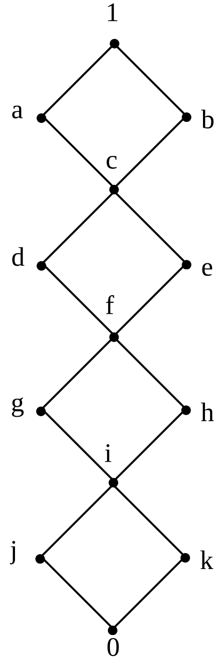
Next we have a class of distributive lattices which are Boolean algebras.

THEOREM 6.4: *Every Boolean algebra of order greater than or equal to four is a Smar lattice.*

***Proof*:** Proof follows from the fact every Boolean algebra B is isomorphic with $\{0, 1\} \times \{0, 1\}$ and in this product if we take T $= \{0, 1\} \times \{0, 1\}$ then T is isomorphic with B the Boolean algebra of order four.

Hence every Boolean algebra is a Smar lattice.

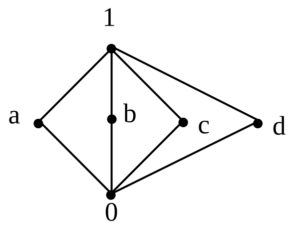
***Example 6.7*:** Let L be a lattice given by the following figure. L is a Smar lattice.



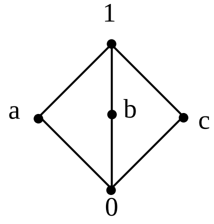
Now we proceed onto define the new notion of Smarandache sublattices and their generalization.

DEFINITION 6.2: *Let L be a lattice. If L has a proper sublattice P and if P is a Smar lattice then we say P is a Smarandache sublattice or Smar sublattice. That is P is not the Boolean algebra of order 4 but contains a Boolean algebra of order four as a sublattice.*

Example 6.8: Let L be lattice given by the following figure.



Take P =



a sublattice of L. P is a Smar lattice. Thus P is a Smar sublattice of L.

THEOREM 6.5: *Let L be a lattice. If L has a proper Smar sublattice P then L is a Smar lattice.*

Proof: Given L is a lattice such that P $\subseteq$ L, P is a proper subset of L and P is a Smar sublattice of L. Let B be a Boolean algebra of order four contained in P, then B is also a subset of L as B $\subseteq$

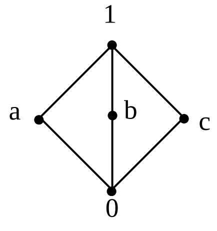
$P \subseteq L$ so L contains a proper subset which is a Boolean algebra of order 4. So L is a Smar lattice.

A natural question would be if L is a Smar lattice will L always contains a Smar sublattice. The answer is not true in general we prove this by the following theorem.

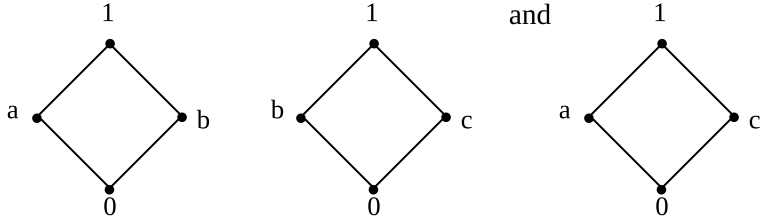
THEOREM 6.6: *Let L be Smar lattice. L in general need not contain a Smar sublattice.*

***Proof*:** To prove the theorem we have to give a Smar lattice which has sublattices but which has no Smar sublattices. Consider the Smar lattice which is a diamond lattice.

Clearly L is Smar lattice as L contains 3 sublattices of order four all of them are isomorphic with the Boolean algebra of order 4.



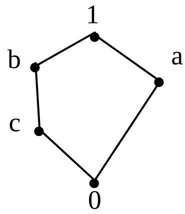
Clearly



are Boolean algebras of order four. But L has no sublattice which is a Smar sublattice as L has no sublattice of order greater than four.

So the diamond lattice is a Smar lattice which has no Smar sublattice but has sublattices.

Likewise consider the pentagon lattice P.



P is a Smar lattice as the sublattices {1, a, b, 0} and {1, a, c, 0} are such that their homomorphic image is isomorphic to the Boolean algebra of order four.

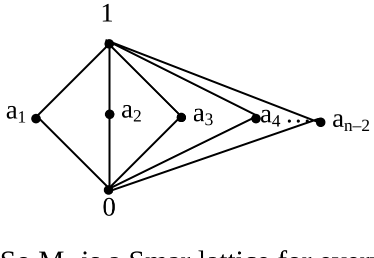
But clearly the pentagon lattice L has no sublattice which is a Smar sublattice of P.

Hence the claim.

If a pertinent to show by some examples that we can have Smar lattices which has Smar sublattices also.

THEOREM 6.7: *Let L be a lattice of the form M_n, n > 5. Then L is a Smar lattice which Smar sublattices.*

***Proof*:** Given M_n is a modular lattice of the form



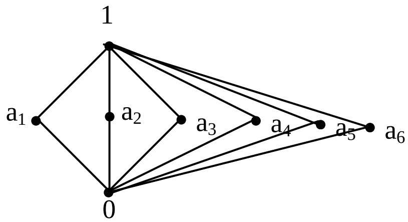
Given n > 5. So M_n is a Smar lattice for every proper subset $P = \{0, 1, a_i, a_{i+1}\} \subseteq M_n$, $1 \leq i \leq n-2$ is a sublattice whose homomorphic image is isomorphic with the Boolean algebra B of order 4.

Thus M_n is a Smar lattice.

Now consider the sublattices P_i of the form $P_i = \{0, 1, a_i, a_{i+1}, a_{i+2}\}$. Each P_i is a Smar sublattice $1 \leq i \leq m$, m a suitable number.

Thus this Smar lattice has Smar sublattices. Thus this class of lattices are Smar lattices which has Smar-sublattices.

***Example 6.9*:** Let M_6 be a lattice given by the following figure.

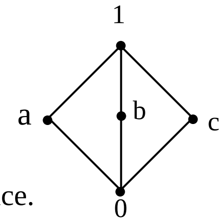


L is a Smar lattice and L has Smar-sublattices.

Now we define those Smar lattices which has no Smar sublattices as simple Smarandache lattices.

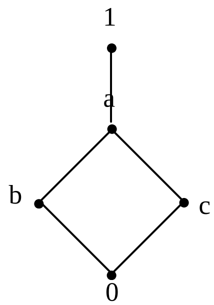
DEFINITION 6.3: *Let L be a lattice which is a Smar lattice if L has sublattices but no Smar sublattices then we call L to be a Smarandache simple lattice (Smar simple lattice) or (Simple Smar lattice).*

***Example 6.10*:** The diamond lattice.



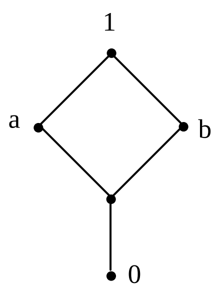
is a simple Smar lattice.

***Example 6.11*:** Let L be the lattice given by the following diagram. L is a simple Smar lattice.



L is a Smar-Lattice which is a simple Smar lattice.

***Example 6.12*:** Let L =



be a lattice given by the figure.

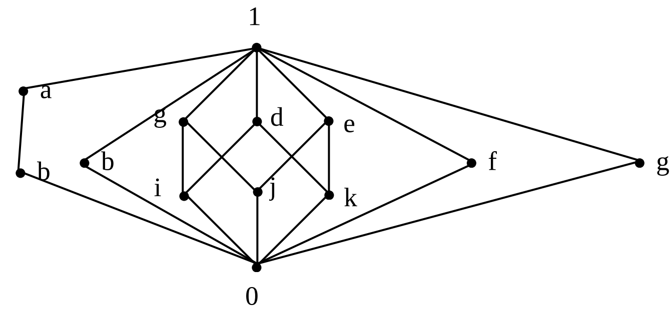
L is a simple Smar lattice.

Next we proceed onto define Smar strong lattices.

DEFINITION 6.4: *Let L be a lattice if L has a sublattice whose homomorphic image is isomorphic to a Boolean algebra of order at least eight then we call L to be a Smarandache strong lattice (S-Smar lattice).*

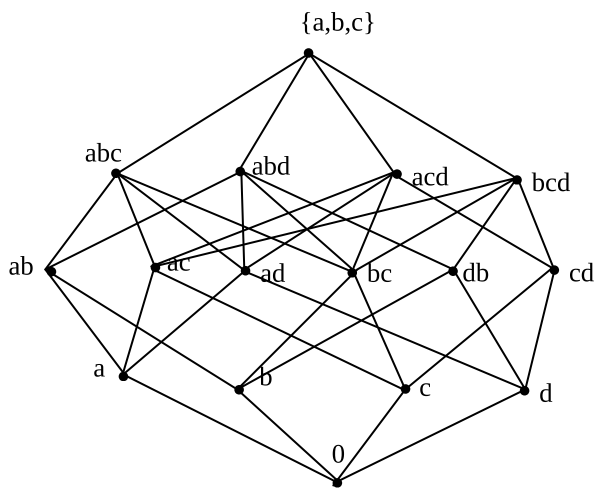
We now proceed onto give a few examples of it.

***Example 6.13*:** Let L be a lattice given by the following figure.



Clearly L is a S-Smar lattice. L is not distributive. L is not modular. L is also a Smar lattice.

***Example 6.14*:** Let L be a lattice given by the following figure.



We see L is a S- Smar lattice (Smar strong lattice).

We have the following interesting.

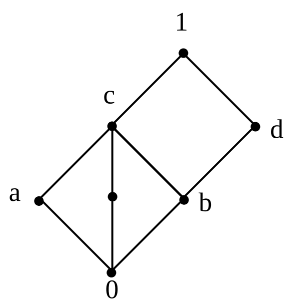
THEOREM 6.8: *Every S-Smar lattice is a Smar lattice but a Smar-Lattice is not in general a S-Smar lattice.*

***Proof*:** Suppose L is a S-Smar lattice (Smar strong lattice) then clearly L has a sublattice. P whose homomorphic image is isomorphic to a Boolean algebra of order 8.

Now every Boolean algebra of order eight has a sub Boolean algebra of order four. Hence L has a sublattice of order four, whose homomphic image is isomorphic with a Boolean algebra of order four. Thus L is a Smar lattice.

Now to show a Smar lattice in general is not a Smar strong lattice we give a example.

Consider a Smar lattice L given by the following figure.



L is a Smar lattice, which is clearly not a Smar strong lattice. Since order of L is itself seven.

Chapter Seven

GB-ALGEBRAIC STRUCTURES

In this chapter we for the first time introduce a new algebraic structure which is not a field or a ring or a near ring or a group ring or a semigroup ring or a lattice or a Boolean algebra or a semiring or a semifield or a vector space or a linear algebra.

This is like a linear algebra having two separate algebraic structures combined in a nice mathematical way.

For constructing this algebraic structure we need a group G and a Boolean algebra B. Just as group ring are defined we define group Boolean algebra and this new algebraic structure is known as GB-algebraic structures.

DEFINITION 7.1: *Let G be a group and B = (B, +, •, 0, 1) be a Boolean algebra. The group Boolean algebra of the group G over the Boolean algebra B consists of all finite formal sums of the form* $\sum_i b_i g_i$ *(i- runs over a finite number) where* $b_i \in B$ *and* $g_i \in G$ *satisfies the following conditions.*

i. $\sum_{i=1}^{n} b_i g_i = \sum_{i=1}^{n} c_i g_i$ *if and only if* $b_i = c_i$ *for* $i = 1, 2, \ldots, n$, $g_i \in G$.

ii. $\sum_{i=1}^{n} b_i g_i + \sum_{i=1}^{n} c_i g_i = \sum_{i=1}^{n} (b_i + c_i) g_i$; $g_i \in G$.

iii. $\left(\sum_{i} b_i g_i\right) \times \left(\sum_{j} \beta_j g_j\right) = \sum_{k} \gamma_k m_k$ *where* $\gamma_k = \sum b_i \beta_j$, $g_i g_j = m_k \in G$ *where* gi, $gj \in G$ *and* $b_i \beta_j$, $\gamma k \in B$.

iv. $b_i g_i = g_i b_i$ *for all* $b_i \in B$ *and* $g_i \in G$.

v. $b \sum_{i=1}^{n} b_i g_i = \sum (bb_i) g_i$ *for* $bb_i \in B$ *and* $\sum b_i g_i \in BG$.

BG is a new special algebraic structure defined as the GB algebraic structure. $0 \in$ B acts as the additive identity of GB. Since $1 \in$ B we have G = 1.G $\subseteq$ BG and B.e = B $\subseteq$ BG where e is in identity element of G.

We denote 1.e = 1.1 = 1 which is the multiplicative identity of BG.

Remark 7.1: We see BG is not any of the known algebraic structures. It is not a Boolean algebra as G $\subseteq$ BG. BG is not a group as BG is not a ring as we do not have inverse elements under addition. So BG is the special GB-algebraic structure.

We give some examples before we proceed on to define more properties about the GB-algebraic structures.

Example 7.1: Let B = {0, 1} be the Boolean algebra of order two. G = $\langle g \mid g^3 = 1 \rangle$ be the cyclic group of order three. BG be the GB-algebraic structure.

$$BG = \{0, 1, g, g^2, 1+g, 1+g^2, \ldots, 1+g+g^2\}.$$

We see $(g + g^2) + (g + g^2) = g + g^2$ as $1 + 1 = 1$ in B.

$$\begin{aligned}(1 + g + g^2) + (g + g^2) &= 1 + (g + g^2) + (g + g^2) \\ &= 1 + g + g^2.\end{aligned}$$

$$g + g^2 + g = g^2 + g$$

(as $g + g = g$)

$$\begin{aligned}(1 + g + g^2)(1 + g) &= 1 + g + g^2 + g + g^2 + 1 \\ &= 1 + g + g^2 \\ g^2 (1 + g) &= g^2 + 1 \text{ and so on.}\end{aligned}$$

We see BG has eight elements and every element x in BG is such that $x + x = x$.

But every element x in BG need not in general be such that $x.x = x$

For take $x = g + g^2$;
$$\begin{aligned}x.x &= (g + g^2)(g + g^2) \\ &= g^2 + 1 + 1 + g \\ &= 1 + g + g^2 \neq g + g^2.\end{aligned}$$

Hence the claim.

Thus BG is not a Boolean algebra.

Further as $(1 + g + g^2)(1 + g + g^2)$ $= 1 + g + g^2$ we see BG is not a group.

Also $0 \in BG$ so BG is not a group.

Example 7.2: Let $B = \{0, 1\}$ be a Boolean algebra of order two and $G = \langle g \mid g^2 = 1\rangle$. Then $BG = \{0, 1, g, 1 + g\}$ be the group Boolean algebra. Here $(1 + g)(1 + g) = 1 + g$ so BG is not a group. As $g.g = 1$, BG is not a Boolean algebra. But $B \subseteq BG$ and $G \subseteq BG$.

Example 7.3: Let B = {0, 1} be the Boolean algebra of order two and G = S_3 be the symmetric group of degree three. Then BG is a GB-algebraic structure where BG = {0, 1, p_1, p_2, p_3, p_4, p_5, 1+p_1, …, 1+p_5, …, p_4 + p_5, …, 1 + p_1 + p_2 + p_3 + p_4 + p_5}. Clearly BG has 2^6 elements. We see their exists x, y $\in$ BG such that xy $\neq$ yx. For p_4. p_2 $\neq$ p_2. p_4 where p_2. p_4 $\in$ BG.

In view of this we make the following definition.

DEFINITION 7.2: *Let BG be a GB-structure where B is a Boolean algebra and G is a group. If in BG for every x, y $\in$ BG we have xy = yx then we call the GB-algebraic structure to be commutative. If BG has atleast one pair of elements x, y such that xy $\neq$ yx then we say the GB-algebraic structure is non commutative.*

The examples 7.1 and 7.2 are commutative GB-algebraic structures where as example 7.3 is a non commutative GB-algebraic structure.

We give the following interesting theorem.

THEOREM 7.1: *Let B be any Boolean algebra. BG the GB-algebraic structure is commutative if and only if G is a commutative group.*

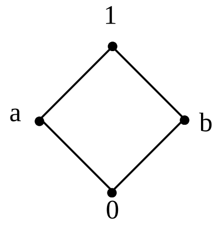
***Proof*:** Given BG is the GB-algebraic structure so B $\subseteq$ BG and G $\subseteq$ BG where B is the Boolean algebra and G is a group. Suppose BG is commutative then we know G $\subseteq$ BG so G is a commutative group.

Now if G is a commutative group then we see from the very definition of the GB-algebraic structure. GB is a commutative algebraic structure.

Corollary 7.1: GB is a non commutative algebraic structure if and only if G is a non commutative group.

***Proof*:** Similar to the theorem 7.1.

***Example 7.4*:** Let B = {0, a, b, 1} be the Boolean algebra of four elements.



a+b = 1 a.b = 0 a.a = a b.b = b.

Let $G = \langle g \mid g^2 = 1\rangle$ be the cyclic group of order two. BG = {0, 1, a, b, g, ag, bg, 1 + g, a + g, b + g, 1 + ag, 1 + bg, b + ag, bg + a, a + ag, b + bg}.

The GB-algebraic structure is commutative and has 16 elements.

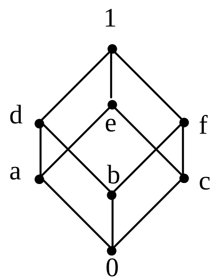
b.ag = 0 and (a+ag) (b+bg) = 0. Thus we see the algebraic structure has zero divisors.

$$\begin{aligned} \text{Also } (a+bg)\,(ag + b) &= a.ag + bg.ag + a.b + b.bg \\ &= ag + 0 + 0 + bg \\ &= (a+b)g = 1.g = g. \end{aligned}$$

Thus it is interesting to see that the group element g in G is got as a product of two distinct elements from BG \ G.

DEFINITION 7.3: *Let BG be the group Boolean algebra (GB-algebraic structure) of the group G over the Boolean algebra B. We say $x \neq 0$ is a zero divisor in BG if there exists a $y \in BG \setminus \{0\}$ such that x.y = 0.*

Example 7.5: Let B = {0, a, b, c, d, e, f, 1} be the Boolean algebra of order 8 given by the following figure.



Let G = ⟨g | g^2 = 1⟩ be the cyclic group of order two. BG = {0, 1, a, b, c, d, e, f, g, ag, bg, cg, dg, eg, fg, …, 1+g, 1+ag, 1+bg, …, 1+fg, a + g, b + g , …, f + g, a + ag, b + bg , …, f + fg, a + bg, a + cg , …, a + fg, b + ag, b + cg , …, b + fg, c + ag, c + bg , …, c + fg, d + ag, d + bg , …, d + fg, e + ag, e + bg , …, e + fg, f + ag, f + bg , …, f + fg}, o(BG) = 64 = 8^2.

We see BG has zero divisors and idempotents.

We prove the following interesting theorem.

THEOREM 7.2: *Let G be any finite group. B = {0, 1} be the Boolean algebra of order two. BG has no zero divisors but has non trivial idempotents.*

***Proof*:** Since B = {0, 1} has no elements x, y such that x.y = 0, x, y ∈ B \ {0} we see in BG no element x ≠ 0 in BG has a y in BG \ {0} such that x.y = 0.

Further $1 + g + g^2 + \ldots + g^{t-1} \in$ BG where t / o(G) and is such that $g^t = 1$ is an idempotent in BG for $(1 + g + \ldots + g^{t-1})(1 + g + \ldots + g^{t-1}) = 1 + g + \ldots + g^{t-1}$ as 1 + 1 = 1 in B.

Hence the claim.

Example 7.6: Let B = {0, 1} be the Boolean algebra of order two and G = ⟨g | g^9 = 1⟩ be cyclic group of order nine. Now BG be the GB-algebraic structure. Take $x = 1 + g + \ldots + g^8$ in BG $x^2 = (1 + g + \ldots + g^8)^2 = (1 + g + \ldots + g^8)$.

Further $x.y \neq 0$ for any $x, y \in BG \setminus \{0\}$

Also that $(1 + x^3 + x^6)(1 + x^3 + x^6)$

$$= 1 + x^3 + x^6 + x^3 + x^6 + 1 + x^6 + 1 + x^3$$
$$= 1 + x^3 + x^6.$$

Hence the claim.

$(1 + x + x^2)(1 + x + x^2)$

$$= 1 + x + x^2 + x + x^2 + x^3 + x^2 + x^3 + x^4$$
$$= 1 + x + x^2 + x^3 + x^4.$$

THEOREM 7.3: *Let G be a group of finite order. B = {0, 1} be the Boolean algebra of order two. BG has non trivial idempotents.*

***Proof*:** Let $o(G) = n$ and $G = \{1, g_1, g_2, \ldots, g_{n-1})$ where each g_i is distinct (i.e., $g_i = g_j$ and only if $i = j$, $1 \leq i, j \leq n-1$). Take $x = 1 + g_1 + \ldots + g_{n-1}$ in BG. We see $x^2 = (1 + g_1 + \ldots + g_{n-1})^2 = 1 + g_1 + \ldots + g_{n-1}$ i.e., x is an idempotent in BG hence the claim.

Example 7.7: Let $G = S_3$ be the symmetric group of degree three group and $B = \{0, 1\}$ be the Boolean algebra of order 2. BG be the GB-algebraic structure.

$x = 1 + p_1 + \ldots + p_5$ in BG is such that $x^2 = x$.

Example 7.8: Let $G = D_{2.6} = \{a.b \mid a^2 = b^6 = 1, bab = a\} = \{1, a, b, b^2, \ldots, b^5, ab, ab^2, ab^3, ab^4, ab^5\}$ be the dihedral group of order six. $B = \{0, 1\}$ be the Boolean algebra of order two. BG be the GB-algebraic structure.

Take $x = 1 + b + b^2 + \ldots + b^5 \in BG$; $x^2 = x$; $y = 1+ab^2 \in BG$ is such that $y^2 = (1+ab^2)^2$

$$= 1 + ab^2 + ab^2 + ab^2ab^2$$
$$= 1+ab^2 + ab^2 + 1$$
$$= 1 + ab^2.$$

BG has atleast 8 idempotents. However it can be easily verified that BG has no zero divisors.

Example 7.9: Let G = S_3 be the symmetric group of degree three i.e., G = S_3 = {1, p_1, p_2, …, p_5} and B = {0, a, b, 1} be the Boolean algebra of order four. BG the GB-algebraic structure. BG has both non trivial zero divisors and nontrivial idempotents.

Take $(ap_1 + ap_2)\ (bp_1 + bp_2)$
$= ab\ p_1^2 + abp_1p_2 + ab\,p_2^2 + abp_2p_1$
$= 0 + 0 + 0 + 0.$

Thus $ap_1 + ap_2$, $bp_1 + bb_2 \in$ BG is a zero divisor in BG.
Also $(1 + p_4 + p_5)\ (1 + p_4 + p_5)$
$= 1 + p_4 + p_5$ in BG.

$(a + bp_1 + cp_2)\ (b + cp_1 + ap_2)$
$= 0 + bp_1 + 0.p_2 + 0.p_1 + 0.p_1 + cp_2\ p_1 + ap_2 + 0\ p_1p_2 + 0\ p_2^2$
$= bp_1 + ap_2 + cp_4 \in$ BG.

We see every product in BG does not in general lead to zero divisor in BG.

Now we proceed onto define GB-sub algebraic structures.

DEFINITION 7.4: *Let G be a group and B be a Boolean algebra. We say BG the group Boolean algebra has a GB-sub algebraic structure S if S contains a proper subset H such that H is a subgroup of G and a proper subset T in S such that T = B or T is a subBoolean algebra of B or if S contains G as a proper subset but B $\subseteq$ S only a proper subset T of S which is a sub Boolean algebra of B; i.e., T $\subseteq$ B and T $\neq$ B.*

We illustrate this by the following examples.

Example 7.10: Let B = {0, 1} be the Boolean algebra of order two. G = $D_{2.6}$ be the dihedral group of order 12. BG be the GB-algebraic structure. Take S = $\{0, 1, b, b^2, b^3, b^4, b^5, 1+b, 1+b^2, 1+b^3, 1+b^4, 1+b^5, b+b^2, b+b^3, b+b^4, b+b^5, b^2+b^3, b^2+b^4, b^2+b^5, b^3+b^4, b^4+b^5, b^3+b^5, 1+b+b^2, \ldots, 1+b+b^2+b^3+b^4+b^5\} \subseteq BG$ is a sub GB-algebraic structure as $B = \{0, 1\} \subseteq S$ and $H = \{1, b, b^2, \ldots, b^5\} \subseteq S$. Infact BG has several GB-sub algebraic structures, like $T_1 = \{0, 1, ab, 1+ab\}$, $T_2 = \{0, 1, a, 1+a\}$ and $T_i = \{1, ab_i, 0, 1+ab_i; 1 \leq i \leq 5\}$; i = 1, 2, …, 5.

THEOREM 7.4: *Let $G = \langle g \mid g^p = 1\rangle$ be a cyclic group of prime order and B = {0, 1} be the Boolean algebra of order two. Then BG the GB-algebraic structure has no proper GB algebraic sub structures.*

***Proof*:** Given G is a cyclic group of prime power order, so G has no proper subgroup. Further B = {0, 1} is a Boolean algebra of order two so B has no proper subalgebra.

So the GB-algebraic structure BG has no proper GB-sub algebraic structures. Thus we have a large class of GB-algebraic structures which has no GB-algebraic sub structures, given by BG where B = {0, 1} and G a cyclic group of prime order.

We illustrate this situation by the following examples.

Example 7.11: Let $G = \langle g \mid g^7 = 1\rangle$ be a cyclic group of order 7 and B = {0, 1} be a Boolean algebra of order two. $BG = \{0, 1, g, \ldots, g^6, 1+g, 1+g^2, \ldots, 1+g+g^2 + g^3 + g^4 + g^5 + g^6\}$ be the group Boolean algebra of B over G. BG has no proper GB-algebraic sub structure.

Another natural question would be can we characterize BG-algebraic structures which has proper sub GB-algebraic structure. The answer is yes which is given in the form of the following theorem.

THEOREM 7.5: *Let G be a group having proper subgroups and B be any Boolean algebra. Then BG the GB-algebraic structure has proper GB-algebraic substructure.*

***Proof*:** Given G has proper subgroups. BG be the GB-algebraic structure. Let H ≠ {e} be proper subgroup of G. BH is a GB-algebraic structure and clearly BH ⊂ BG.

Hence the claim.

THEOREM 7.6: *Let G be any group. B a Boolean algebra of order greater than or equal to four. Then BG the GB-algebraic structure has proper GB-algebraic substructures.*

***Proof*:** Given G is any group and B a Boolean algebra of order greater than or equal to four, so B contains T = {0, 1} as a proper Boolean subalgebra.

Now TG is a GB-algebraic substructure which is properly contained in BG; hence the claim.

Example 7.12: Let B = {0, a, b, 1} be a Boolean algebra, G = A_4, the alternating subgroup of the symmetric group S_4. BG is the GB-algebraic structure. Now take T = {0, 1}, TA_4 is a GB-algebraic structure contained in BG. So BG has GB-algebraic substructure.

Example 7.13: Let G = $\langle g \mid g^{17} = 1\rangle$ be the group of order 17. B = {0, a, b, 1} be the Boolean algebra of order four. BG be the GB-algebraic structure. Take T = {0, 1}, TB is a GB-algebraic structure such that TB ⊆ GB so BG has GB-algebraic substructures.

Now we define a new notion.

DEFINITION 7.5: *Let B be a Boolean algebra and G be any group. BG be the GB-algebraic structure. If BG has no GB-algebraic substructure then we call BG to be a simple GB-algebraic structure.*

We have a class of simple GB algebraic structures. Take B = {0, 1} to be the Boolean algebra of order two and

$G = \langle g \mid g^p = 1\rangle$; p a prime; BG the group Boolean algebra is a simple GB-algebraic structure.

We also have a class of GB-algebraic structure which are not simple GB-algebraic structure. Take G be a group of any finite order n, n a composite number. B be any Boolean algebra then BG the GB-algebraic structure is not a simple GB-algebraic structure.

Now we give some more illustrations.

Example 7.14: Let $G = \langle g \mid g^{31} = 1\rangle$ and $B = \{0, 1\}$ be the group of order 31 and the Boolean algebra of order two respectively. Then BG the GB-algebraic structure is simple. Further BG has no zero divisors. $o(BG) = 2^{31}$.

Also $(1 + g + \ldots + g^{31})^2 = (1 + g + \ldots + g^{31})$ i.e., BG has no trivial idempotents. Now take $(1+g) . (g^2 + g^4) = g^2 + g^4 + g^3 + g^5$ where $1 + g, g^2 + g^4 \in BG$.

Take $g + g^2$ and $g^2 + g^7$ in BG. Consider $(1 + g^2) (g^2 + g^7) = g^2 + g^4 + g^9 + g^7$.

Suppose $1 + g + g^2 + g^3 + g^7, g^2 + g^5 + g + g^{11} \in BG$.

Consider their product

$(1 + g + g^2 + g^3 + g^7) (g^2 + g^5 + g + g^{11}) = g^2 + g^3 + g^4 + g^5 + g^9 + g^5 + g^6 + g^7 + g^8 + g^{12} + g + g^2 + g^3 + g^4 + g^8 + g^{11} + g^{12} + g^{13} + g^{14} + g^{18} = g + g^2 + g^3 + g^4 + g^5 + g^6 + g^7 + g^8 + g^9 + g^{11} + g^{12} + g^{13} + g^{14} + g^{18}$.

Now we define one more new concept.

DEFINITION 7.6: *Let* $BG = \left\{ \sum_i b_i g_i \,\middle|\, b_i \in B \text{ and } g_i \in G\right\}$ *be the GB-algebraic structure of the group G over the Boolean algebra B.*

Let $\alpha = \sum b_g g \in BG$ we define content of α denoted by $cont\alpha = \{g \in G \mid b_g \neq 0\}$. Clearly $cont\alpha$ is a finite subset of G.

If we have α in BG to be central in BG and if $x \in$ cont α. If $y \in G$ then $x^y = y^{-1} xy \in$ cont $y^{-1} xy \in$ cont $y^{-1} \alpha y =$ contα.

Since cont α is finite we have only a finite number of distinct x^y with $y \in G$.

It is interesting to study the set of elements $x \in G$ with this property.

We see for $\alpha\ \beta \in BG$ where $\alpha = \sum b_g g$ and $\beta = \sum c_h h$, h, g $\in G$ and b_g, $c_h \in B$ the cardinality of cont α and cont β is such that $|\text{cont } \alpha\beta| \leq |\text{cont } \alpha|\ |\text{cont } \beta|$.

We first illustrate this by some simple examples.

Example 7.15: Let $G = S_3 = \{1, p_1, p_2, p_3, p_4, p_5\}$ be the symmetric group of degree three and $B = \{0, 1\}$ be the Boolean algebra of order two. BG the GB-algebraic structure.

Let $\alpha = \{p_1 + p_2 + p_3\}$ and $\beta = 1 + p_4$ be in BG.

We see content of $\alpha = \{p_1, p_2, p_3\}$ and
content of $\beta = \{1, p_4\}$.

$$\begin{aligned} \text{Now } \alpha\beta &= (p_1 + p_2 + p_3)(1 + p_4) \\ &= p_1 + p_2 + p_3 + p_3 + p_1 + p_2 \\ &= p_1 + p_2 + p_3. \end{aligned}$$

Thus content of $\alpha\beta = \{p_1, p_2, p_3\}$ and $|\text{cont } \alpha\beta| = 3$ but $|\text{cont } \alpha|\ |\text{cont } \beta| = 3.2 = 6$.

Hence we see $|\text{cont } \alpha\beta| \leq |\text{cont } \alpha|\ |\text{cont } \beta|$.

Example 7.16: Let $G = \langle g \mid g^{10} = 1\rangle$ be the cyclic group of order 10 and B = {0, 1} be the Boolean algebra of order two BG be the GB-algebraic structure.

$BG = \{0, 1, g, g^2, \ldots, g^9, 1+g, \ldots, 1+g^9, g+g^2, \ldots, g^8+g^9, \ldots, 1+g + \ldots g^9\}$.

Take $\alpha = \{1 + g + g^2 + g^3 + g^4\}$ and $\beta = \{g + g^5 + g^6 + g^7 + g^8\}$ in BG.

Now cont $\alpha = \{1, g, g^2, g^3, g^4\}$
$|\text{cont } \alpha| = 5$. cont $\beta = \{g, g^5, g^6, g^7, g^8\}$; $|\text{cont } \beta| = 5$

$$\begin{aligned}
\alpha\beta &= (1 + g + g^2 + g^3 + g^4) (g + g^5 + g^6 + g^7 + g^8) \\
&= g + g^2 + g^3 + g^4 + g^5 + g^6 + g^7 + g^8 + g^9 + g^6 + g^7 + g^8 + g^9 + 1 + g^7 + g^8 + g^9 + 1 + g + g^8 + g^9 + 1 + g + g^2 \\
&= 1 + g + g^2 + g^3 + g^4 + g^5 + g^6 + g^7 + g^8 + g^9.
\end{aligned}$$

Cont $\alpha\beta = \{1, g, g^2, \ldots, g^9\}$ and $|\text{cont } \alpha\beta| = 10$.

Thus $|\text{cont } \alpha\beta| \leq |\text{cont } \alpha|\, |\text{cont } \beta|$.

Example 7.17: Let $G = \{g \mid g^6 = 1\}$ be the cyclic group of order 6 and B = {0, a, b, 1} be the Boolean algebra of order four. $BG = \{0, a, b, 1, g, g^2, \ldots, g^5, ag, ag^2, \ldots, ag^5, bg, bg^2, \ldots, bg^5, \ldots, 1 + g + g^2 + g^3 + g^4 + g^5, a + ag + ag^2 + ag^3 + ag^4 + ag^5, b + bg + bg^3 + bg^4 + ag^5\}$ be the GB-algebraic structure.

Take $\alpha = a + ag + \ldots + ag^5$ and $\beta = b + bg + \ldots + bg^5$ in BG.

Cont $\alpha = \{1, g, g^2, g^3, g^4, g^5\} = G$ and $|\text{cont } \alpha| = |G| = 6$.

Cont $\beta = \{1, g, g^2, \ldots, g^5\} = G$ and $|\text{cont } \beta| = |G| = 6$.

We find $\alpha\beta$.

$\alpha\beta$ = $(a + ag + \ldots + ag^5)$ $(b + bg + \ldots + bg^5)$ = a $(1 + g + \ldots + g^5)$ b $(1 + g + \ldots + g^5)$ = ab $(1 + g + \ldots + g^5)^2$ = ab $(1+g + g^2 + \ldots + g^5) = 0$ as a.b = 0 in B.

Thus $\alpha\beta = \phi$, $|\text{cont } \alpha\beta| = 0$.

We see $|\text{cont } \alpha\beta| \leq |\text{cont } \alpha|\ |\text{cont } \beta|$.

Let $\alpha = (1 + g^2 + g^4)$ and $\beta = (b + bg^2 + bg^4)$ in BG we see
Cont $\alpha = \{1, g^2, g^4\}$ and $|\text{cont } \alpha| = 3$.
Cont $\beta = \{1, g^2, g^4\}$ and $|\text{cont } \beta| = 3$.

$$\begin{aligned} \alpha\beta &= (1 + g^2 + g^4)\ (b^1 + bg^2 + bg^4) \\ &= b + bg^2 + bg^4 + bg^2 + bg^4 + bg^4 + bg^6 + bg^6\ (g^6 = 1) \\ &= b + bg^2 + bg^4 \end{aligned}$$

Cont $\alpha\beta = \{1, g^2, g^4\}$ and $|\text{cont } \alpha\beta| = 3$
Thus $|\text{cont } \alpha\beta| \leq |\text{cont } \alpha|\ |\text{cont } \beta|$.

We see $\alpha\beta = \beta$ in this case. We cannot say $\alpha\beta = \beta$ can be simplified as $\alpha\beta - \beta = 0$ or $(\alpha-1)\beta = 0$ as in GB-algebraic structure the notion of $-\alpha$ for any $\alpha \in$ BG does not have any meaning as B is a Boolean algebra.

Now before we prove $|\text{ cont } \alpha\beta| \leq |\text{cont } \alpha|\ |\text{cont } \beta|$ we first prove in case of finite groups G we have $|\text{cont } \alpha|\ |\text{cont } \beta| \leq |G|\ |G|$ and $|\text{cont } \alpha\beta|$ can be 0.

Thus we have $0 \leq |\text{cont } \alpha\beta| \leq |\text{cont } \alpha|\ |\text{cont } \beta| \leq |G|\ |G|$.

THEOREM 7.7: *Let G be any finite group. B = {0, 1} be the Boolean algebra of order two. BG the GB-algebraic structure. Let $\alpha, \beta \in$ BG be such that cont $\alpha \neq \phi$ and cont $\beta \neq \phi$ then cont $\alpha\beta \neq \phi$,*

i.e., if $|\text{cont } \alpha| \neq \phi$ and $|\text{cont } \beta| \neq \phi$ then $|\text{cont } \alpha\beta| \neq \phi$.

***Proof* :** Given B = {0, 1} is a Boolean algebra of order two and G a finite group with α, $\beta \in$ BG such that |cont α| $\neq \phi$ and |cont β| $\neq \phi$. Now in cont $\alpha\beta$ we see no term can vanish as 0, as B has no zero divisor coefficient terms in α and β (i.e., cont α and cont β remain as non zero). So cont $\alpha\beta \neq \{0\}$.

Hence |cont $\alpha\beta$| $\neq \phi$ in case B is a boolean algebra of order two.

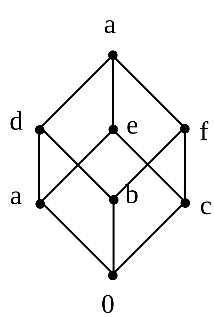
Now we cannot assert in this way in case of Boolean algebras of order greater than two for these Boolean algebras B always have complements of elements in B such that their product is always zero.

We first illustrate this by an example before we prove any result in this direction.

Example 7.18: Let G = $\langle g \mid g^8 = 1\rangle$ be the cyclic group of order eight and B = {0, a, b, c, d, e, f, 1} be a Boolean algebra of order eight.

Let BG be the GB-algebraic structure. We can have α, $\beta \in$ BG with |cont α| $\neq 0$ and |cont β| $\neq 0$ but |cont $\alpha\beta$| = 0.

For take B to be the Boolean algebra given by the figure.



We have a.b = 0 = b.c = a.c = d.c = e.b = f.a.

Let α = ag + a and $\beta = bg^2 + bg + bg^3 + fg^5 + f$ be in BG.

Clearly cont α = {1, g}, |cont α| = 2, cont β = {1, g, g^2, g^3, g^5} and |cont β| = 5.

Now $\alpha\beta = (ag + a)(bg^2 + bg + bg^3 + fg^5 + f) = abg^4 + abg^2 + bag^2 + abg + abg^4 + abg^3 + afg^6 + afg^5 + afg + af = 0$ as ab = fa = 0.

Cont $\alpha\beta = \phi$ so |cont $\alpha\beta$| = 0.

Thus in the case of Boolean algebras of order greater than two we can have |cont $\alpha\beta$| = 0 without |cont α| = 0 and |cont β| = 0.

We see some more interesting properties about cont α, $\alpha \in$ BG.

Let α, $\beta \in$ G where BG is a GB-algebraic structure |cont α| + |cont β| $\geq$ |cont $(\alpha+\beta)$|.

We first illustrate this by the following examples.

Example 7.19: Let G be the finite dihedral group $D_{2.6}$ and B = {0, 1} be the Boolean algebra of order two. BG be the GB-algebraic structure.

$D_{2.6}$ = {1, a, b, b^2, …, b^5, ab, ab^2, …, ab^5} be the given Dihedral group.

BG = $\{\sum \alpha_i g_i;\ \alpha_i \in B$ and $g_i \in D_{2.6}\}$ be the GB-algebraic structure.

Let $\alpha = a + b + ab^3 + ab^5 + b^2$ and $\beta = a + ab^2 + ab^3 + b^2 + b^5 + b^3$ be elements of BG.

Now contα = {a, b, ab^3, b^2, ab^5} so |contα| = 5 and contβ = {a, ab^2, ab^3, b^2, b^5, b^3, b} so that |contβ| = 7.

Now $\alpha+\beta = (a + b + ab^3 + ab^5 + b^2 + a + ab^2 + ab^3 + b^2 + b^5 + b^3 + b$
$= a + b + ab^2 + ab^3 + b^2 + b^3 + ab^5$.

cont $(\alpha+\beta) = \{a, b, ab^2, ab^3, b^2, b^3, b^5, ab^5\}$ and
$|\text{cont } (\alpha+\beta)| = 8$.

$|\text{cont } (\alpha+\beta)| \leq |\text{cont } \alpha| + |\text{cont } \beta|$.

Example 7.20: Let $G = \langle g \mid g^{11} = 1\rangle$ be the group of order 11 and $B = \{0, a, b, 1\}$ be the Boolean algebra of order four. BG be the GB-algebraic structure.

Let $\alpha = (1 + g + g^{10} + g^2 + bg^3 + g^4 + ag^5 + bg^6 + g^7 + g^8 + bg^9)$ and

$\beta = (a + bg + g^2 + ag^3 + bg^4 + g^5 + ag^6 + bg^7 + g^8 + bg^9 + ag^{10})$ be take from BG. Now cont$\alpha = \{1, g, g^{10}, g^2, g^3, g^4, g^5, g^6, g^8, g^7, g^9\}$ cont$\beta = \{1, g, g^2, g^3, g^4, \ldots, g^{10}\}$.

We see $|\text{cont } \alpha| = |\text{cont } \beta| = 11 = |G|$.

Now $\alpha+\beta = (1 + g + g^2 + bg^3 + g^4 + ag^5 + bg^6 + g^7 + g^8 + bg^9 + ag^{10})$

$(a + bg + g^2 + ag^3 + bg^4 + g^5 + ag^6 + bg^7 + g^8 + bg^9 + ag^{10}) = (1+a) + (a+b)g + (1+1)g^2 + (b+a)g^3 + (1+b)g^4 + (a+1)g^5 + (b+a)g^6 + (1+b)g^7 + (a+1)g^8 + (b+b)g^9 + (a+a)g^{10})$

$= 1 + g + g^2 + g^3 + g^4 + g^5 + g^6 + g^7 + g^8 + bg^9 + ag^{10}$.

Cont $\alpha+\beta = \{1, g, g^2, g^3, \ldots, g^{10}\}$

Thus $|\text{cont } \alpha+\beta| < |\text{cont } \alpha| + |\text{cont } \beta|$.

Now it is interesting to see $|\text{cont } \alpha+\beta| \neq 0$ if $|\text{cont } \alpha| \neq 0$ (or) $|\text{cont } \beta| \neq 0$; for in a Boolean algebra B if $a, b \in B \setminus \{0\}$; $a+b \neq 0$.

We further see $|\text{cont}\ (\alpha+\beta)|$ is always less than or equal to order of G.

Thus $o(G) \leq |\text{cont}\ (\alpha+\beta)| \leq |\text{cont}\ \alpha| + |\text{cont}\ \beta| \leq 2\ o(G) = 2|G|$.

The proof of the above inequality is left as an exercise for the reader.

Now we proceed onto define the notion of invariant and universally invariant elements of the GB-algebraic structure.

DEFINITION 7.7: *Let G be any group of B be any Boolean BG the GB-algebraic structure. $\alpha \in BG$ is said to be invariant in BG if there exists a $\beta \in BG \setminus \{0\}$ such that $\alpha.\beta = \alpha$.*

If $\alpha \in BG$ is such that $\alpha.\beta = \alpha$ for all $\beta \in BG \setminus \{0\}$ then we say α is an universally invariant element of BG.

It is interesting to see $0 \in BG$ is an universally invariant element of BG.

THEOREM 7.8: *Let BG be a GB-algebraic structure where $B = \{0, 1\}$ is a Boolean algebra of order two and G any group.*

If $\alpha \in BG$ is universally invariant element of BG, then α is an invariant element of BG. Further an invariant element of BG in general is not an universally invariant element of BG.

***Proof*:** From the very definition of the universally invariant and invariant element of BG we see every universally invariant element of BG is an invariant element of BG.

To prove that converse we give an example.

Take $G = D_{2.9} = \{1, a, b \mid a^2 = b^9 = 1, bab = a\}$ to be the dihedral group of order 18 and $B = \{0, 1\}$ the Boolean algebra of order two. BG be the GB-algebraic structure.

Take $\alpha = 1 + b + b^2 + \ldots + b^8$ in BG. Now there is a $\beta = 1 + b^3 + b^6 \in$ BG such that $\alpha\beta = \alpha$.

Take $\gamma = a + b$ in BG.

Now $\alpha\gamma = (1 + b + b^2 + b^3 + \ldots + b^8)\ (a + b)$

$= \quad a + ba + b^2a + b^3a + \ldots + b^8a + b + b^2 + b^3 + b^4 + \ldots + b^8 + 1.$

Clearly $\alpha\beta \neq \alpha$. Thus α is not a universally invariant element of BG.

Take $\alpha = \{1 + a + b + \ldots + ab^8 + ab + ab^2 + \ldots + ab^8\}$ in BG. We see $\alpha\beta = \alpha$ for every β = BG \ {0}. This α is a universally invariant element of BG. $0 \in$ BG is also universally invariant element of BG as $0.\alpha = 0$ for every $\alpha \in$ BG.

It is pertinent to mention here that when B is any other Boolean algebra of higher order say ($|B| \geq 4$) then we cannot say BG will contain universally invariant elements other than zero. However we may have invariant elements.

Example 7.21: Let $G = \langle g \mid g^4 = 1\rangle$ be the cyclic group of order four. B = {0, a, b, 1} be the Boolean algebra of order four. Let BG be the GB-algebraic structure. Let $\alpha = 1 + g + g^2 + g^3$. α is an invariant element of BG but α is not an universally invariant element of BG for it we take $\beta = ag + bg^2$.

We see $\alpha\beta \neq \alpha$. But
$\alpha.\ (g+g^2) = \alpha$

$\alpha(1+g) = \alpha$ and
$\alpha(1+g+g^2) = \alpha$ and so on.

The following problems are left open for an innovative reader to solve.

Problem 1: If $\alpha \in B$ is an universally invariant element of BG (where B = {0, 1}) and G is a finite group does it imply cont α is order of G. (G a finite group).

Problem 2: Is it possible for BG to have a universally invariant elements if $|B| \geq 4$?

THEOREM 7.9: *Let G be an abelian group such that H is a proper finite subgroup of G.*

B = {0, 1} be a Boolean algebra of order 2. Let BG be the GB-algebraic structure. Then $\alpha \in BG$ such that contα = H is a invariant element of BG.

***Proof*:** Let H be the finite subgroup of G, say $H = \{1, h_1, \ldots, h_t\} \subseteq G$. Take $\alpha = (1 + h_1 + \ldots + h_t) \in$ BG is an invariant element of G as $\alpha.\beta = \alpha$ for every $\beta \in BG \setminus \{0\})$ with cont $\beta \subseteq H$.

Hence the claim.

Now we are interested in studying whether the GB-algebraic structure BG contains any other universally invariant elements other than $\alpha = 1 + g_1 + \ldots + g_n$ where $G = \{1, g_1, \ldots, g_n\}$ and B = {0, 1}. ($0 \in$ BG is trivially universally invariant element of BG as 0. $\alpha = 0$ for all $\alpha \in$ BG).

Example 7.22: Let $G = D_{2.7} = \{a, b \mid a^2 = b^7 = 1; bab = a\}$ be the dihedral group of order 14. B = {0, 1} be the Boolean algebra of order two BG the GB algebraic structure BG has atleast one universally invariant element and atleast eight invariant elements.

Let $x = \sum_{ix} g_i \in$ BG with cont α = G. Then α is a universally invariant element of BG. Let $S_1 = 1 + a$, $S_2 = 1+ab$, $S_3 = 1 + ab^2$, $S_4 = 1+ab^3$, $S_5 = 1 + ab^4$, $S_6 = 1 + ab^5$, $S_7 = 1 + ab^6$

and $S_8 = 1 + b + b^2 + b^3 + b^4 + b^5 + b^6$ we see $(1+a)a = 1+a = S_1$ so is invariant.

$(1+ab)\ (ab) = 1 + ab$ so S_2 is invariant. Likewise we can show these eight element in BG are invariant elements of BG. It is left for the reader to find other invariant elements of BG.

DEFINITION 7.8: *Let G be a non commutative group and B any Boolean algebra. Let BG be the GB-algebraic structure. We define center of BG denoted by C(BG) = $\{x \in BG \,/\, x\,\alpha = \alpha\, x$ for all $\alpha \in BG\}$.*

Example 7.23: Let G = S_3 be the symmetric group of degree three and B = {0, 1} be the Boolean algebra of order two.

$C(BG) = \{0, 1, 1+p_1 + p_2 + p_3 + p_4 + p_5, 1 + p_4 + p_5, p_1 + p_2 + p_3, 1 + p_1 + p_2 + p_3, p_1 + p_2 + p_3 + p_4 + p_5, p_4 + p_5\}$

We see $o(BG) = 2^6$ where as $o(C(BG)) = 2^3$. Further C(BG is a GB-subalgebraic structure of BG.

Let G be a commutative group B any Boolean algebra BG be the GB-algebraic structure. Clearly C(BG) = BG we know if G is a non commutative group then $C(BG) \subseteq BG$.

Now we find out whether C(BG) has any nice algebraic structure. To be more precise. Can C(BG) be a GB-algebraic substructure of BG?

In view of this we define the following new notion.

DEFINITION 7.9: *Let G be any group and B a Boolean algebra. BG the GB-algebraic structure. Let P be a proper subset of the BG. We say P is a GB-algebraic pseudo substructure if the following condition are true.*

(1) $0, 1 \in P$,
(2) for every $\alpha, \beta \in P$; $\alpha + \beta \in P$,
(3) for every $\alpha, \beta \in P$; $\alpha\beta$ and $\beta\alpha \in P$.

THEOREM 7.10: *Let G be a group. B a Boolean algebra. BG be the GB-algebraic structure. C(BG) be the center of BG. C(BG) is a GB-algebraic pseudo substructure of BG.*

***Proof*:** Let C(BG) = $\{\alpha \in BG / x\,\alpha = \alpha\, x$ for all $x \in BG\}$ be the center of BG. To prove C(BG) is a GB-algebraic pseudo substructure. Clearly $0, 1 \in C(BG)$.

Take $\alpha, \beta \in C(BG)$; to show $\alpha + \beta \in C(BG)$. Given $\alpha\, x = x\,\alpha$ and $\beta\, x = x\,\beta$ for all $x \in BG$.

$$(\alpha+\beta)\, x = \alpha x + \beta x = x\,\alpha + x\,\beta$$
$$= x(\alpha+\beta). \text{ So } \alpha + \beta \in C(BG)$$

Let $\alpha, \beta \in C(BG)$ to show $\alpha\beta \in C(BG)$.

Now to show $(\alpha\beta)x = x\,(\alpha\beta)$ for all $x \in BG$.

$$(\alpha\beta)x = \alpha(\beta x) = \alpha(x\beta)$$
$$= (\alpha x)\beta = (x\alpha)\beta = x\,(\alpha\beta)$$

Thus $\alpha\beta \in C(BG)$.

Thus C(BG) is a GB-algebraic pseudo substructure of BG.

Example 7.24: Let B = {0, 1} be the Boolean algebra of order two. A_4 be the alternating subgroup of S_4. BA_4 be the GB-algebraic structure. To find $C(BA_4)$. Clearly $0, 1 \in C\,(BA_4)$.

Take $\alpha = 1 + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} +$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \in BA_4.$$

We see $\alpha \in CB(A_4)$.

Let $\beta = \sum_{g_i} g_i \in BA_4$ such that cont $\beta = A_4$, then $\beta \in (BA_4)$.

Thus $C(BA_4) = \{0, 1, \alpha, \beta\}$ we see $C(BA_4)$ is a GB-algebraic pseudo substructure of BA_4.

Thus we have proved C(BG) is only a GB-algebraic pseudo substructure of BG.

It is pertinent to mention here that we have no relation between the GB-algebraic substructure and GB-algebraic pseudo substructure. For we see in case of GB-algebraic pseudo substructure P, we do not have a subgroup of G to be a subset of P. We have of course the two element Boolean algebra to be always a subset of P. Of course we may have a GB-algebraic substructure to be a proper subset of a GB-algebraic pseudo substructure and vice versa. To this end we give some examples.

Example 7.25: Let $G = D_{2.8} = \{a, b \mid a^2 = b^8 = 1; bab = a\}$ be the dihedral group. $B = \{0, 1\}$ be the Boolean algebra. BG is the GB-algebraic structure. Now $C(BD_{2.8}) = \{0, 1, b^4, \alpha = \sum g_i$, (cont $\alpha = D_{2.8}$), $\beta = 1 + b^4\}$ is a GB-algebraic pseudo substructure of BG.

Clearly $C(BD_{2.8})$ is a GB-algebraic substructure for as it contains GB-algebraic substructure namely $T = \{0,1, b^4, 1 + b^4\} \subseteq C(BD_{28})$, $B = \{0, 1\}$ and $H = \{1, b^4\}$.

So study of conditions for a GB-algebraic pseudo substructure to be also a GB-algebraic substructure is an interesting problem.

Example 7.26: Let $G = S_3 \times D_{2.7}$ be a group and $B = \{0, 1\}$ be a Boolean algebra. We see $H = \{(1, p_1) \times D_{2.7} \subseteq G$ be the subgroup of G. BH is a GB-algebraic substructure of BG but BH is not a GB-algebraic pseudo substructure of BG.

Example 7.27: Let $G = S_3 \times A_4 \times D_{2.7}$ be the group and $B = \{0, 1\}$ be the Boolean algebra of order two. Let BG be the GB-algebraic structure.

Let $H = S_3 \times \{e\} \times \{1\} \subseteq G$; then BH is the GB-algebraic substructure of BG. For $H \subseteq G$ is a subgroup and $B = \{0, 1\}$ is the Boolean algebra. If we take $K = S_3 \times \{e\} \times D_{2.7}$ the subgroup of G then $BK \subseteq BG$ is the GB algebraic substructure of BG.

We can take $T = \{e\} \times \{e\} \times D_{2.7} \subseteq G$ then BT is the GB-algebraic substructure of BG.

Now having seen substructures of BG we now proceed onto define the notion of the BG normalized subalgebraic structure.

DEFINITION 7.10: *Let G be any group and B any Boolean algebra. BG the GB-algebraic structure. Suppose $T \subseteq BG$ be a proper GB-algebraic substructure of BG such that*

$gTg^{-1} \subseteq T$ for every $g \in G$ then we call T to be a normalized GB-algebraic substructure of BG or T is called the normal GB-algebraic substructure of BG.

We illustrate this by some examples.

Example 7.28: Let $G = A_4$ be the alternating subgroup of S_4. $B = \{0, 1\}$ be the Boolean algebra. BG be the GB-algebraic structure.

$$\text{Take } H = \left\{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}\right\} \subseteq A_4.$$

$B \subseteq BH$ is a GB-algebraic normal substructure of BG. Clearly $g(BH)g^{-1} \subseteq BH$ for every $g \in A_4$.

We prove the following important and interesting theorems.

THEOREM 7.11: *Let G be a group having normal subgroups. B any Boolean algebra, BG the GB-algebraic structure. If H is a normal subgroup of G then BH is a GB-algebraic normal substructure of BG.*

***Proof*:** Given G is a group and B any Boolean algebra. BG the GB-algebraic structure. Let H be a nontrivial normal subgroup of H. BH is clearly a GB-algebraic substructure of BG. We see for every $g \in G$, $gBHg^{-1} \subseteq BH$ as $gHg^{-1} = H$. Hence BH is a GB-algebraic normal substructure of BG.

THEOREM 7.12: *Let G be a simple group (i.e., G has no nontrivial normal subgroups. B the Boolean algebra of order two then BG has no GB- normal algebraic substructure.*

***Proof*:** If G is a simple group it clearly implies G has no normal subgroups. We see it T is any GB-algebraic substructure then T should contain a proper subset H which is a subgroup of G and or a proper subset $B_1 \subseteq T$ where B_1 is a Boolean subalgebra of G. Since B is a Boolean algebra of order two we cannot find $B_1 \subseteq B$. So the only condition for T to be a GB-algebraic substructure is we need T to contain a proper subset H which is a subgroup of G.

Thus if T is to be a GB-algebraic normal substructure we need $gTg^{-1} \subseteq T$ this would be possible only if the subgroup H of G is normal in G.

Example 7.29: Let $G = S_3 \times A_4$ be a group $B = \{0, 1\}$ be a Boolean algebra. BG be the GB-algebraic structure.

$$\text{Let H} = \left\{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}\right\} \subseteq G$$

be a subgroup of G. We see BH is a GB-algebraic substructure of BG but BH is not a GB-algebraic normal substructure of BG.

However it view of this we give the following results.

THEOREM 7.13: *Let G be a simple group. B a Boolean algebra of order two. BG the GB-algebraic structure. BG has GB-algebraic substructure which is not a GB-algebraic normal substructure.*

Proof: Without loss of generally we can assume BG is the GB-algebraic structure in which the group G has subgroups which are not normal in G. Let H be a subgroup of G which is not a normal subgroup of G. We see BH is a GB-algebraic substructure which is clearly not a GB-algebraic normal substructure.

THEOREM 7.14: *Let G = $\langle g \mid g^p = 1\rangle$ where G is a cyclic group of order p, p a prime B = {0, 1} be a Boolean algebra of order two we see BG, the GB-algebraic structure has no proper GB-algebraic substructures and no proper GB-algebraic normal substructures.*

Proof: Given G is a cyclic group of order p, p a prime and B = {0, 1} be the Boolean algebra of order two. BG is a GB-algebraic structure. If H is to be any proper GB-algebraic substructure of BG we must have in H a proper subset P in H such that P is a proper subgroup of G. But G has no proper subgroups, hence H cannot be a GB-algebraic substructure of BG.

Hence the claim.

THEOREM 7.15: *Let G be a simple group. B be a Boolean algebra of order greater than or equal to four. Then the GB-algebraic structure has GB-algebraic normal substructures.*

***Proof*:** Given B is a Boolean algebra of order greater than or equal to four. So B has $B_1 = \{0, 1\}$ to be sub Boolean algebra of B. Take B_1G, B_1G is a GB-algebraic substructure of BG and B_1G is such that $gB_1Gg^{-1} = B_1G$ for every $g \in G$.

Hence the claim.

Now having seen some of the properties of GB-algebraic substructure we illustrate them by some examples.

Example 7.30: Let $G = \langle g \mid g^p = 1\rangle$ be a cyclic group of prime order p and $B_1 = \{0, 1, a, b\}$ be a Boolean algebra of order four. B_1G is a GB-algebraic structure. Take $B = \{0, 1\} \subseteq B_1$ clearly BG is a GB-algebraic normal substructure of BG.

Hence the claim.

Example 7.31: Let $G = D_{2p} = \langle a, b \,/\, a^2 = b^p = 1, bab = a\rangle$ be the dihedral group of order 2p where p is a prime. $B = \{0, 1\}$ be the Boolean algebra of order two. BG the GB-algebraic structure has GB-algebraic normal substructures.

$H = \{1, b, b_2, \ldots, b_{p-1}\}$ be normal subgroup of $G = D_{2p}$ so BH is a GB algebraic normal substructure of BG.

Example 7.32: Let S_n be a symmetric group of degree n, $B = \{0, 1\}$ be the Boolean algebra of order two. BS_n be the GB-algebraic structure. BS_n has GB-algebraic normal substructure. Take in BA_n, A_n the normal subgroup of S_n. BA_n is a GB-algebraic substructure of BS_n. Hence BA_n is a GB algebraic normal substructures of BS_n.

THEOREM 7.16: *Let A_n be a alternating group ($n \geq 5$ or $n = 3$). Let $B = \{0, 1\}$ be a Boolean algebra of order two. BA_n has no GB-algebraic normal substructures.*

Proof**:** The result follows from the fact that A_n is simple $n \geq 5$ or $n = 3$. Hence BA_n has no GB-algebraic normal substructures.

Corollary: Let A_4 be the alternating group. $B = \{0, 1\}$ be the Boolean algebra of order four. Then BA_4 has GB-algebraic normal subsubstructures.

Proof**:** Since A_4 has H =

$$\left\{\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 3 & 4\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix}\right\}$$

to be normal subgroup of A_4 we have BH to be GB-algebraic normal substructures of BA_4.

Further take $K = \left\{\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 3 & 4\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 3 & 4 & 2\end{pmatrix},\right.$

$$\left.\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 4 & 2 & 3\end{pmatrix}\right\} \subseteq A_4;$$

BK is only GB-algebraic substructure of BA_4 and is not a GB-algebraic normal substructure of BA_4.

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INDEX

L

M

N

P

S

Z

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On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.
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In this book the authors introduce a new class of lattces called super modular lattices which is an equational class of lattices lying between the equational class of distributive lattices and modular lattices. Several other new properties related with these lattices are introduced, described and developed.

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